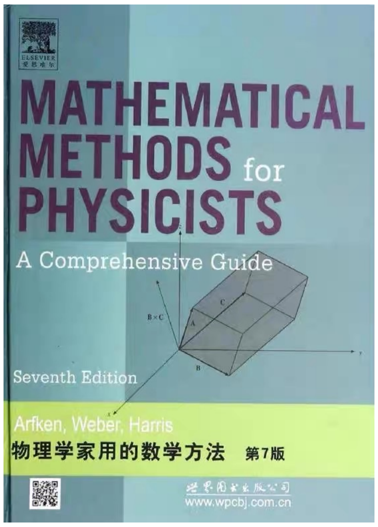
物理数学（分析基础选讲）



b站：泰山讨论班

网站：physicsseminar.com

1. Infinite series

1.1 $S_i = \sum_{n=1}^{i} u_n$

$\{S_i\}$, $\lim_{n\to\infty} S_n = S$, 收敛

必要条件：$\lim_{n\to\infty}(S_n - S_{n-1}) = S - S = 0$

$\quad\quad\quad\quad \|$

$\quad\quad \lim_{n\to\infty} u_n$

Cauchy criterion: $\forall \varepsilon > 0,\ \exists N$, s.t. $\forall\, i, j > N$, $|S_j - S_i| < \varepsilon$

Example 1.

$1 + r + r^2 + \cdots$

$S_n = \dfrac{1 - r^n}{1 - r}$

if $|r| < 1$, $S = \dfrac{1}{1-r}$

$r \geq 1$, S 发散

Example 2.

$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$

$= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots$

$> 1 + \frac{1}{2} + \frac{1}{2} + \cdots \to \infty$

1.2 Comparison Test.

1.2.1 $0 \le u_n \le \underline{a_n}$ - - - n^{-p} $p > 1$

$0 \le \underline{b_n} \le u_n$ - - - n^{-p} $p \le 1$

1.2.2. Cauchy Root Test.

$a_n \le r^n$

For large n, $\underline{(a_n)^{1/n} \le r} < 1$, 收敛.

$(a_n)^{1/n} \geq 1$, 发散.

$\lim_{n\to\infty} (a_n)^{1/n} < 1$ 收敛

$\lim_{n\to\infty} (a_n)^{1/n} > 1$ 发散

$\lim_{n\to\infty} (a_n)^{1/n} = 1$ 不确定.

保序性

$a_n \le b_n \Rightarrow \lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n$

$\lim_{n\to\infty} a_n < \lim_{n\to\infty} b_n \Rightarrow a_n < b$

1.2.3 D'Alembert Ratio Test.

For large n, $a_{n+1}/a_n \le r < 1$, 收敛

$a_{n+1}/a_n \geq 1$, 发散.

$n > N$

$a_{N+1}/a_N \le r$

$a_{N+1} \le a_N \underline{r}$

$a_{N+2} \le a_N r^2$

$\vdots$

Example 3.

$$\sum_{n=1}^{\infty} \frac{n}{2^n}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \cdots < 1$$

1.3. More sensitive Test

1.3.1. Kummer's theorem, $\sum_n u_n$

$$\lim_{n\to\infty}\left(a_n \frac{u_n}{u_{n+1}} - a_{n+1}\right) \geq c > 0 \Rightarrow \sum_n u_n \text{ 收敛}.$$

$$\lim_{n\to\infty}\left(a_n \frac{u_n}{u_{n+1}} - a_{n+1}\right) \leq 0, \text{且} \sum_n a_n^{-1} \text{ 发散} \Rightarrow \sum_n u_n \text{ 发散}.$$

若 $\sum_n a_n^{-1}$ 收敛，

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} > \lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$$

若 $\sum_n a_n^{-1}$ 发散，依然成立，且 $\sum_n a_n^{-1}$ 发散得越慢，判别越强

N，

$$u_{N+1} \leq \left(a_N u_N - \cancel{a_{N+1} u_{N+1}}\right)/c$$

$$u_{N+2} \leq \left(\cancel{a_{N+1} u_{N+1}} - \cancel{a_{N+2} u_{N+2}}\right)/c \Rightarrow \sum_{i=N+1}^{n} u_i \leq \frac{a_N u_N}{c} - \frac{a_n u_n}{c}$$

$$\cdots$$

$$u_n \leq \left(\cancel{a_{n-1} u_{n-1}} - a_n u_n\right)/c \qquad < \frac{a_N u_N}{c}$$

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① D'Alembert Test 取 $a_n = 1$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} > 1 \quad \text{收敛}$$

② Raabe Test 取 $a_n = n$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) > 1 \quad \text{收敛}$$

③ Bertrand's Test 取 $a_n = n \ln n$

$$\lim_{n\to\infty}\left(n \ln n \frac{u_n}{u_{n+1}} - (n+1)\ln(n+1)\right) > 0 \quad \text{收敛}$$

④ Extended Bertrand's Test，取 $a_n = n \prod_{k=1}^{N} \ln_{(k)} n$

1.3.2 Gauss' Test

For large n, $\frac{u_n}{u_{n+1}} = 1 + \frac{h}{n} + \frac{B(n)}{n^2}$

in which $B(n)$ is bouded,

if $h>1$, $\sum_n u_n$ is Converged,

$h \le 1$, $\sum_n u_n$ is diverged.

Confirm it, take $a_n = n \ln n$.

$$\lim_{n\to\infty}\left[n\ln n\left(1+\frac{h}{n}+\frac{B(n)}{n^2}\right) - (n+1)\ln(n+1)\right]$$

$$= \lim_{n\to\infty}\left[(n+1)\ln n + (h-1)\ln n + \frac{B(n)\ln n}{n} - (n+1)\ln(n+1)\right]$$

$$= \lim_{n\to\infty}\left[-(n+1)\ln\frac{n+1}{n} + (h-1)\ln n\right]$$

$h \le 1$，发散

$h > 1$，收敛

Example 4.

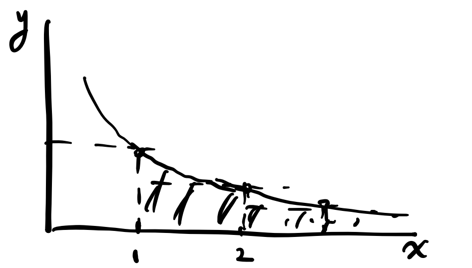
$$\frac{a_{2j+2}}{a_{2j}} = \frac{2j(2j+1)-\lambda}{(2j+1)(2j+2)}$$

$$[2j(2j+1)-\lambda]^{-1} = [2j(2j+1)]^{-1}\left(1+\frac{\lambda}{2j(2j+1)}\right)$$

$$\lim_{j\to\infty}\frac{u_j}{u_{j+1}} = \lim_{j\to\infty}\frac{(2j+1)(2j+2)}{2j(2j+1)-\lambda} = \frac{\cancel{(2j+1)}(2j+2)}{2j\cancel{(2j+1)}}\left(1+\frac{\lambda}{2j(2j+1)}\right)$$

$$= \frac{2j+2}{2j} + \frac{4\lambda/(2j+1)}{j^2} = 1 + \frac{1}{j} + \frac{4\lambda/(2j+1)}{j^2}$$，发散.

1.3.3. Cauchy Integral Test.



$f(n) = a_n$, $f(x)$ 连续、单减.

$$s_i = \sum_{n=1}^{i} a_n = \sum_{n=1}^{i} f(n)$$

$$s_i \geq \int_1^{i+1} f(x)\,dx$$

$$s_i - a_1 \leq \int_1^{i} f(x)\,dx \Rightarrow \int_1^\infty f(x)\,dx \leq \sum_{n=1}^\infty a_n \leq \int_1^\infty f(x)\,dx + a_1$$

估计余项 (remainder)

$$\sum_{n=1}^\infty a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^\infty a_n$$

$$\Rightarrow \int_{N+1}^\infty f(x)\,dx \leq \sum_{n=N+1}^\infty a_n \leq \int_{N+1}^\infty f(x)\,dx + a_{N+1}$$

只需 $f(x)$ 连续

$$\sum_{n=N_1+1}^{N_2} f(x) = \int_{N_1}^{N_2} f(x)\,dx + \int_{N_1}^{N_2} (x - \lfloor x \rfloor) f'(x)\,dx$$

Proof:

$$\int_{N_1}^{N_2} x f'(x)\,dx = N_2 f(N_2) - N_1 f(N_1) - \int_{N_1}^{N_2} f(x)\,dx$$

$$\int_{N_1}^{N_2} \lfloor x \rfloor f'(x)\,dx = \sum_{n=N_1}^{N_2-1} \int_n^{n+1} f'(x)\,dx = \sum_{n=N_1}^{N_2-1} n\,[f(n+1) - f(n)]$$

$$= -\sum_{n=N_1+1}^{N_2} f(n) - N_1 f(N_1) + N_2 f(N_2)$$

Example 5.

$$\zeta(p) = \sum_{n=1}^\infty n^{-p} \ , \quad \int_1^\infty x^{-p}\,dx = \begin{cases} \dfrac{x^{-p+1}}{-p+1}\Big|_1^\infty & p \neq 1 \\ \ln x \Big|_1^\infty & p = 1 \end{cases}$$

$$\gamma = \lim_{n\to\infty} \left(\sum_{m=1}^{n} m^{-1} - \ln n \right)$$ Euler constant.

Example 6.

$$S = \sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

$$\int_2^{\infty} \frac{1}{n \ln n} dn = \int_2^{\infty} \frac{1}{\ln n} d(\ln n) = \ln(\ln n)\Big|_2^{+\infty}$$

1.4 Alternating Series. $\sum_n (-1)^{n+1} a_n$

1.4.1. Leibniz criterion, if a_n monotonically decreasing, $\lim_n a_n = 0$,

$\sum_n (-1)^{n+1} a_n$ 收敛.

$$R_{2n} = (a_{2n+1} - a_{2n+2}) + (a_{2n+3} - a_{2n+4}) + \cdots$$

$$= a_{2n+1} - (a_{2n-2} - a_{2n-3}) - (a_{2n+4} - a_{2n+5}) + \cdots$$

$\Rightarrow \quad 0 < R_{2n} < a_{2n+1} \qquad$ error < 0

$R_{2n+2} < R_{2n+1} < 0 \qquad$ error > 0

Example 7.

$$S = \sum_{n=1}^{\infty} \frac{\cos(nx)}{n}, \quad \text{(Dirichlet's Test).}$$

$$S = \int_1^{\infty} \frac{\cos nx}{n} dn + \int_1^{\infty} (n - \lfloor n \rfloor)\left[-\frac{x}{n}\sin(nx) - \frac{\cos nx}{n^2}\right] dn$$

$$= \left[\frac{\sin nx}{nx}\right]_1^{\infty} + \frac{1}{x}\int_1^{\infty} \frac{\sin nx}{n^2} dn - \int_1^{\infty} (n - \lfloor n \rfloor) \frac{\cos nx}{n^2} dn$$

$- x\int_1^{\infty} (n - \lfloor n \rfloor) \frac{\sin nx}{n} dn$, Let $(n - \lfloor n \rfloor)\sin nx = g'(n)$,

$g(N) = \int_1^N (n - \lfloor n \rfloor) \sin nx \, dn$,

$$\Rightarrow \int_1^{\infty} \frac{g'(n)}{n} dn = \left[\frac{g(n)}{n}\right]_1^{\infty} + \int_1^{\infty} \frac{g(n)}{n^2} dn, \text{ 收敛}$$

Evaluate $\sum_{n=1}^{\infty} \frac{\cos nx}{n} = Re\left[\sum_{n=1}^{\infty} \frac{e^{inx}}{n}\right]$

$f(x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots = -\ln(1-x)$

$f'(x) = 1 + x + x^2 + \cdots = \frac{1}{1-x}$

$e^{ix} = \cos x + i\sin x$

$Re\, z = (z+\bar{z})\frac{1}{2}$

$\overline{f(i)} = f(-i)$

$\Rightarrow$ 原式 $= Re[-\ln(1-e^{ix})]$

$= [-\ln(1-e^{ix}) - \ln(1-e^{-ix})]\frac{1}{2}$

$= -\frac{1}{2}\ln(1+1-e^{ix}-e^{-ix})$

$= -\frac{1}{2}\ln(2-2\cos x) = -\ln(2\sin\frac{x}{2})$

$\cos x = \frac{e^{ix}+e^{-ix}}{2}$

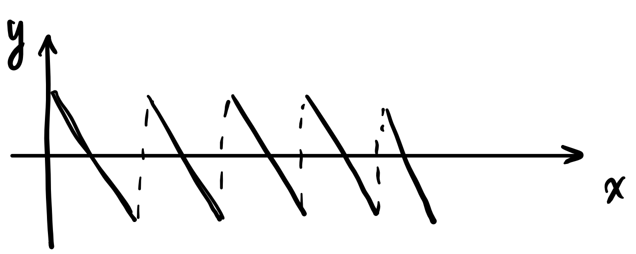
$\sin x = \frac{e^{ix}-e^{-ix}}{2i}$

$\sum_{n=1}^{\infty}\frac{\sin nx}{n} = Im\left[\sum_{n=1}^{\infty}\frac{e^{inx}}{n}\right] = Im[-\ln(1-e^{ix})]$

$= [-\ln(1-e^{ix}) + \ln(1-e^{-ix})]\frac{1}{2i}$

$= \frac{1}{2i}\ln\left(\frac{1-e^{-ix}}{1-e^{ix}}\right) = \frac{1}{2i}\ln(-e^{-ix}) = \frac{1}{2i}\ln\left(e^{i(\pi-x+2k\pi)}\right)$

$= \frac{1}{2}(\pi - x + 2k\pi)$



$\sum_n a_n b_n$

Abel. $\{b_n\}$单调有界, $\sum_{n=1}^{\infty} a_n$ 收敛 $\Rightarrow \sum_n a_n b_n$ 收敛.

Dirichlet. $\{b_n\} \xrightarrow{单调} 0$, $\{a_n\}$部分和有界 $\Rightarrow \sum_n a_n b_n$ 收敛.

1.5 Absolute / Conditional Converge

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots = \ln 2$$

$$\frac{1}{2}S = \frac{1}{2} \quad - \frac{1}{4} \quad + \frac{1}{6} \quad - \frac{1}{8} + \cdots$$

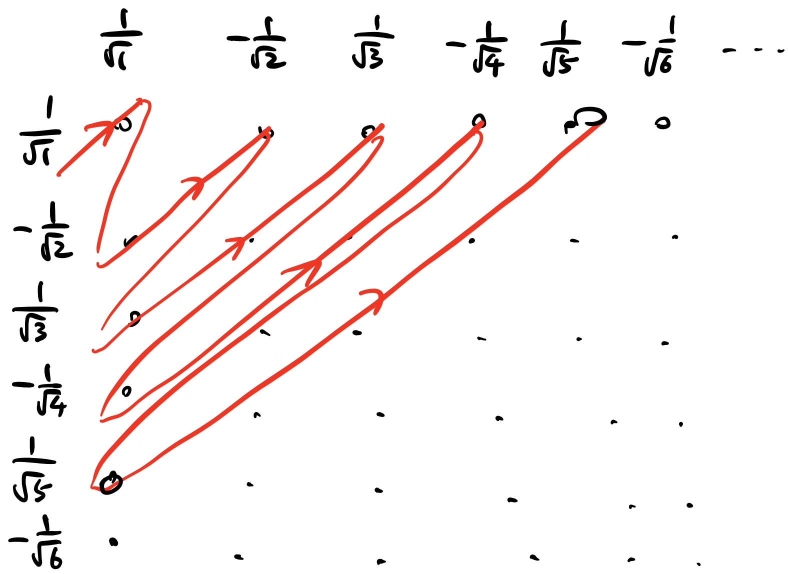
$$\frac{3}{2}S = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots = \frac{3}{2}\ln 2$$

Riemann's Theorem.

Example 8.

$$\left[\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}\right]^2$$

$$= \sum_{n=1}^{\infty} (-1)^n \left[\frac{1}{\sqrt{1}}\frac{1}{\sqrt{n-1}} + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{n-2}} + \cdots + \frac{1}{\sqrt{n-1}}\frac{1}{\sqrt{1}}\right]$$



1.6. Improvement of convergence

$$\alpha_1 = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

$$\alpha_2 = \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4}$$

$$- - - - -$$

$$\alpha_p = \sum_{n=1}^{\infty} \frac{1}{n \cdots (n+p)} = \frac{1}{p\, p!}$$

$$\zeta(3) = \sum_n \frac{1}{n^3}$$

$$\zeta(3) + a\alpha_2$$

$$= \sum_n \left(\frac{1}{n^3} + \frac{a}{n(n+1)(n+2)}\right)$$

$$= \sum_n \frac{\cancel{n^2(1+a)} + 3n+2}{n^3(n+1)(n+2)}$$

取 $a = -1$

$$\zeta(3) = \sum_n \frac{3n+2}{n^3(n+1)(n+2)} + \frac{1}{4}$$

Example 9.

$$\sum_n \frac{1}{1+n^2} = \sum_n n^{-2}(n^{-2}+1)^{-1} = \sum_n n^{-2}\left(\frac{1+n^{-2}-n^{-2}+n^{-4}-n^{-4}+n^{-6}-n^{-6}}{1+n^{-2}}\right)$$

$$= \sum_n n^{-2}\left(1-n^{-2}+n^{-4}-\frac{n^{-6}}{1+n^{-2}}\right)$$

$$= \zeta(2)-\zeta(4)+\zeta(6)-\sum_{n=1}^{\infty}\frac{1}{n^8+n^6}$$

1.7 Series of function

$$\sum_{n=1}^{\infty} u_n(x) = S(x) = \lim_{n\to\infty} S_n(x)$$

1.7.1 Uniform convergence

$\forall \varepsilon>0$, $\exists N$, $\forall n>N$, $\forall x\in[a,b]$, s.t. $|S(x)-S_n(x)|<\varepsilon$.

Example 10 $[0,1]$

$$S(x) = \sum_{n=0}^{\infty}(1-x)x^n = \begin{cases} 1 & 0\le x<1 \\ 0 & x=0 \end{cases}$$

$\forall x\neq 0$, $S_N = 1-x^N$

$$|1-(1-x^N)| = x^N < \varepsilon$$

1.7.2 Weirestrass M Test (优势级数)

$\forall x\in[a,b]$, $u_i(x)\le M_i$

$\sum_i M_i$ 收敛 $\Rightarrow$ $\sum_n u_i(x)$ 一致收敛.

1.7.3 Abel、Dirichlet Test.

Example 11.

$$S(x) = \sum_{n=1}^{\infty}\frac{(-1)^n}{n+x^2} \qquad |S(x)-S_n(x)| < |u_{n+1}(x)| \le |u_{n+1}(0)|$$

$S(x)$ is uniformly convergent, but not absolutely convergent.

1.7.4. Properties

$\sum_n u_n$ 一致收敛，$x \in [a,b]$，$u_n(x)$ 连续.

1. $S(x) = \sum_n u_n(x)$ 连续

2. $\int_a^b S(x)dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)dx$

3. $\frac{d}{dx} S(x) = \sum_{n=1}^{\infty} \frac{d}{dx} u_n(x)$

$$\left(\begin{array}{l} \frac{d}{dx} u_n(x) \quad \text{连续} \\ \sum_n \frac{d}{dx} u_n(x) \quad \text{一致连续} \end{array} \right)$$

1.8. Taylor series

$$f(x+h) = \sum_{n=0}^{\infty} \frac{h^n}{n!} f^{(n)}(x) = \sum_{n=0}^{\infty} \frac{h^n D^n}{n!} f(x) = e^{hD} f(x)$$

令 $D = \frac{d}{dx}$

1.8.1. Radius of convergence

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = R^{-1}$$

$\forall\ 0 < S < R$, $\forall\ x \in [-S, S]$, it is AC and UC.

1.8.2 Uniqueness Theorem

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$$

$$\Rightarrow \quad a_n = b_n$$

1.9. Binomial Theorem

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2}x^2 + \cdots + R_n \qquad -1 < x < 1$$

$$R_n = \frac{x^n}{n!}(1+\xi)^{m-n} m(m-1)\cdots(m-n+1)$$

$$\binom{m}{n} = \frac{m(m-1)\cdots(m-n+1)}{n!} = \frac{m!}{n!\,(n-m)!}$$

$$(1+x)^m = \sum_{n=0}^{\infty}\binom{m}{n}x^n$$

$\forall m<0, \quad m=-p$

$$\binom{-p}{n} = \frac{(-p)(-p-1)\cdots(-p-n+1)}{n!} = (-1)^n\frac{p(p+1)\cdots(p+n-1)}{n!}$$

$$= (-1)^n\frac{(p+n-1)!}{n!\,(p-1)!}$$

Pochhammar symbol $(a)_n$

$$(a)_0 = 1, \quad (a)_1 = a, \quad (a)_{n+1} = a(a+1)\cdots(a+n)$$

$$\binom{m}{n} = \frac{(m-n+1)_n}{n!}$$

$$(1+x)^{-1/2}$$

$$\binom{-\frac{1}{2}}{n} = \frac{1}{n!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2n-1}{2}\right) = (-1)^n\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n\cdot n!}$$

$$= (-1)^n\frac{(2n-1)!!}{(2n)!!}$$

$$0!! = (-1)!! = 1$$

Example 12.

$$y = \sin^{-1} x .$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \Rightarrow \sin^{-1} x = \int_0^x \frac{dt}{\sqrt{1-t^2}} = \int_0^x \sum_{n=0}^{\infty} (-1)^n \frac{(2n-1)!!}{(2n)!!} (-t^2)^n \, dt$$

$$= \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} \frac{x^{2n+1}}{2n+1}$$

$$(a_1 + \cdots + a_m)^n = \sum_{n_1 + \cdots + n_m = n} \frac{n!}{n_1! \, n_2! \cdots n_m!} a_1^{n_1} a_2^{n_2} \cdots a_m^{n_m} .$$

1.10 Mathematical Induction

1. $n=1$, $(n=k) \Rightarrow (n=k+1)$,

2. $n=1$, $(n=1,2,\cdots,k) \Rightarrow (n=k+1)$.

Example 13. $1 + \alpha_1 x + \alpha_2 x^2 + \cdots$

$$(1+\alpha_1 x) \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} + a_1 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{n+1}}{n}$$

$$= x + \sum_{n=2}^{\infty} (-1)^{n-1} \left(\frac{1}{n} - \frac{a_1}{n-1} \right) x^n$$

$$= x + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{(1-a_1)n - 1}{n(n-1)} x^n$$

令 $a_1 = 1$,

$$\ln(1+x) = \left(\frac{x}{1+x} \right) \left(1 - \sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)} x^n \right)$$

1.11. $$f(x) = \sum_{n=0}^{\infty} (-1)^n c_n x^n = \frac{1}{1+x} \sum_{n=0}^{\infty} (-1)^n a_n \left(\frac{x}{1+x} \right)^n$$

$a_0 = c_0$, $a_1 = c_1 - c_0$, $a_2 = c_2 - 2c_1 + c_0$, ...

$$a_n = \sum_{j=0}^{n} (-1)^j \binom{n}{j} c_{n-j}$$

Example 14.

$$\frac{\ln(1+x)}{x} = 1 - \frac{x}{2} + \frac{x^3}{3} - \cdots$$

$$a_n = \frac{(-1)^n}{n+1}$$

$$\frac{\ln(1+x)}{x} = \frac{1}{1+x}\left[1 + \frac{1}{2}\left(\frac{x}{1+x}\right) + \frac{1}{3}\left(\frac{x}{1+x}\right)^2 + \cdots\right]$$

$$\Rightarrow \ln(1+x) = \left(\frac{x}{1+x}\right) + \frac{1}{2}\left(\frac{x}{1+x}\right)^2 + \frac{1}{3}\left(\frac{x}{1+x}\right)^3 + \cdots \qquad x < \infty$$

2. Vectors and Complex numbers.

2.1. Functions in the complex domain

$$e^z = 1 + z + \frac{1}{2!}z^2 + \cdots$$

$$e^{iz} = 1 + iz + \frac{1}{2!}(iz)^2 + \cdots = \cos z + i\sin z.$$

2.2. Polar representation.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow z = x + iy = re^{i\theta}$$

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

$$\big||z_1| - |z_2|\big| \le |z_1 + z_2| \le |z_1| + |z_2|$$

$$\mathrm{Re}\, z = \frac{z+\bar{z}}{2}, \quad \mathrm{Im}\, z = \frac{z-\bar{z}}{2}$$

2.3. Circular and hyperbolic function.

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

$$\cosh\theta = \frac{e^{\theta}+e^{-\theta}}{2}, \quad \sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2}$$

$$\cosh iz = \cos z, \quad \cos iz = \cosh z$$

$$\sinh iz = i\sin z, \quad \sin iz = i\sinh z.$$

$$e^{i\theta} = i\sin\theta + \sqrt{1-\sin^2\theta} = i\frac{\tan\theta}{\sqrt{1+\tan^2\theta}} + \frac{1}{\sqrt{1+\tan^2\theta}}$$

$$\sin^{-1}z = -i\ln\left[iz + \sqrt{1-z^2}\right]$$

$$\tan^{-1}z = -i\ln\left[i\frac{z}{\sqrt{1+z^2}} + \frac{1}{\sqrt{1+z^2}}\right]$$

$$= -i\ln\left[\frac{1+zi}{\sqrt{1+zi}\sqrt{1-zi}}\right] = \frac{i}{2}\left[\ln(1-iz) - \ln(1+iz)\right]$$

$$\sinh^{-1}z = \ln(z+\sqrt{1+z^2})$$

$$\tanh^{-1}z = \frac{1}{2}\left[\ln(1+z) - \ln(1-z)\right]$$

2.4. Power and Roots

$$z = re^{i\varphi}, \quad z^n = r^n e^{in\varphi}, \quad z^{1/n} = r^{1/n} e^{i(\varphi+2\pi m)/n}$$

Example, $1^{1/3} = 1, e^{i\frac{2}{3}\pi}, e^{i\frac{4}{3}\pi}$

2.5. Logarithm.

$$\ln z = \ln r + i(\theta + 2\pi m)$$

$$\mathrm{Ln}\, z = \ln r + i\theta$$

2.6. Fail of power and logarithm identity.

$$x^y = e^{y \ln x} \quad \checkmark$$

$$x = \ln e^x \quad \times \quad \rightarrow \quad \ln(-1)^2 \neq 2\ln(-1)$$

$$\mathrm{Ln}(ab) = \mathrm{Ln}\, a + \mathrm{Ln}\, b \quad \times \quad \rightarrow \quad \mathrm{Ln}(-i) = \ln 1 - \frac{i\pi}{2} = -\frac{i\pi}{2}$$

$$\mathrm{Ln}(-1) + \mathrm{Ln}\, i = \frac{3}{2} i\pi$$

$$(a^b)^c = (a^c)^b = a^{bc} \quad \times$$

$$\hookrightarrow \quad ((-1)^2)^{1/2} \neq ((-1)^{1/2})^2$$

1. $\ln w^z \neq z \ln w$

$$w^z = e^{z \ln w} = e^{z[\mathrm{Ln}\, w + i 2\pi n]}$$

$$\{\ln w^z\} = \{ z\mathrm{Ln}\, w + z \cdot 2\pi i \underline{n} + 2\pi i \underline{m} \}$$

$$\{z \ln w\} = \{ z \mathrm{Ln}\, w + z \cdot 2\pi i n \}$$

2. $[(-1)\cdot(-1)]^{1/2} = \{1, -1\}$

3. Derivatives and extrema.

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

3.1. $$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds}$$

3.2. $$\left(\frac{\partial y}{\partial x}\right)_f = -\frac{\left(\frac{\partial f}{\partial x}\right)_y}{\left(\frac{\partial f}{\partial y}\right)_x}$$

3.3. $\frac{\partial f}{d\vec{s}} = 0$, $\forall \vec{s} \neq 0 \iff \frac{\partial f}{\partial x_i} = 0$ Stationary point

$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial x \partial y} > 0$ ✓ $\frac{\partial^2 f}{\partial x^2} < 0$ 极大.

< 0 ×

$= 0$ 不确定.

4. Evaluation of Integrals

1. Special functions.

Gamma $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$

Factorial $n! = \int_0^\infty t^n e^{-t}\, dt$

Riemann Zeta $\zeta(x) = \frac{1}{\Gamma(x)} \int_0^{+\infty} \frac{t^{x-1}\, dt}{e^t - 1}$

expotential Integral $E_n(x) = \int_1^\infty t^{-n} e^{-t}\, dt$

Sine Integral $S_i(x) = -\int_x^{+\infty} \frac{\sin t}{t}\, dt$

Cosine Integral $C_i(x) = -\int_x^{+\infty} \frac{\cos t}{t}\, dt$

Error function $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ $\mathrm{erf}(\infty) = 1$

$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{+\infty} e^{-t^2} dt$ $\mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$

Dilogarithm $Li_2(x) = -\int_0^x \frac{\ln(1-t)}{t}\, dt$

Example. 15.

$$I = \int_0^\infty \frac{e^{-x^2}}{x^2 + a^2}\, dx$$

$$J(t) = \int_0^{+\infty} \frac{e^{-t(x^2+a^2)}}{x^2+a^2}\,dx \quad , \quad I = e^{a^2} J(1)$$

$$J'(t) = -\int_0^{+\infty} e^{-t(x^2+a^2)}\,dx = -e^{-ta^2}\int_0^{+\infty} e^{-tx^2}\,dx$$

$$= -\frac{1}{2}\sqrt{\frac{\pi}{t}}\, e^{-ta^2}$$

$$J(t) = \frac{\sqrt{\pi}}{2}\int_t^{+\infty} \frac{e^{-t'a^2}}{t'^{1/2}}\,dt' = \frac{\sqrt{\pi}}{a}\int_{at^{1/2}}^{\infty} e^{-u^2}\,du$$

$$= \frac{\pi}{2a}\,\mathrm{erfc}(a t^{1/2})$$

$$I = e^{a^2} J(1) = \frac{\pi}{2a} e^{a^2}\,\mathrm{erfc}(a)$$

Example 16.

$$I = \int_0^1 \ln\left(\frac{1+x}{1-x}\right)\frac{dx}{x}$$

$$= \int_0^1 dx \cdot 2\left[1 + \frac{x^2}{3} + \frac{x^4}{5} + \cdots\right] = 2\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

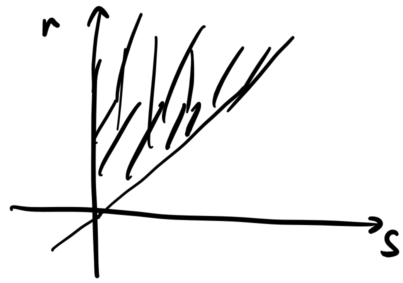
$$\frac{1}{2^2}\zeta(2) = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots \quad \Rightarrow \quad \frac{3}{4}\zeta(2) = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

$$\Rightarrow \quad I = \frac{3}{2}\zeta(2)$$

Example 17.

$$I = \int_0^{+\infty} e^{-at}\cos bt\,dt = \mathrm{Re}\left[\int_0^{+\infty} e^{(-a+ib)t}\,dt\right]$$

$$= \mathrm{Re}\left(\frac{a+ib}{a^2+b^2}\right)$$



Example 18.

$$\int_0^{+\infty} e^{-r}\,dr \int_r^{\infty} \frac{e^{-s}}{s}\,ds$$

$$= \int_0^{\infty} \frac{e^{-s}}{s}\,ds \int_0^{s} e^{-r}\,dr$$

$$= \int_0^{+\infty} \frac{e^{-s}}{s}(1-e^{-s})\,ds$$

$$= \int_0^{\infty} \frac{e^{-s}}{s} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} s^n}{n!}\,ds = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \int_0^{+\infty} s^{n-1} e^{-s}\,ds$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}(n-1)! = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln 2$$

2. $dx_1 \cdots dx_n \longrightarrow dy_1 \cdots dy_n$

$$dx_1 \cdots dx_n = J\, dy_1 \cdots dy_n$$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_2}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_1} \\ \frac{\partial x_1}{\partial y_2} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_2} \\ \vdots & & & \\ \frac{\partial x_1}{\partial y_n} & \cdots & \cdots & \frac{\partial x_n}{\partial y_n} \end{vmatrix} = \frac{\partial(x_1 \cdots x_n)}{\partial(y_1 \cdots y_n)}$$

5. Dirac Delta function

$$\begin{cases} \delta(x) = 0 & x \neq 0 \\ \int_a^b f(x)\,\delta(x)\,dx = f(0) & \Rightarrow \int_{-\infty}^{+\infty} \delta(x)\,dx = 1 \end{cases}$$

5.1.

$$\delta_n = \begin{cases} 0 & x < -\frac{1}{2n} \\ n & -\frac{1}{2n} < x < \frac{1}{2n} \\ 0 & x > \frac{1}{2n} \end{cases}$$

$$\delta_n = \frac{n}{\sqrt{\pi}} e^{-n^2x^2} \quad \left[\int_{-\infty}^{+\infty} e^{-n^2x^2} dx = \sqrt{\frac{\pi}{n}} \right]$$

$$\delta_n = \frac{n}{\pi} \frac{1}{1+n^2x^2}$$

$$\delta_n = \frac{\sin nx}{\pi x} = \frac{1}{2\pi}\int_{-n}^{n} e^{ixt}\,dt$$

$$\delta(x) = \lim_{n\to\infty} \delta_n .$$

5.2. linear functional

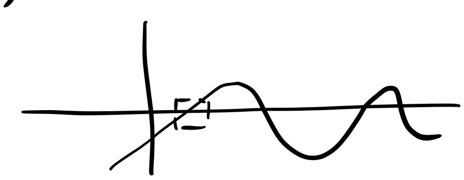
5.3. Properties of $\delta(x)$ (积分意义下)

① $\delta(ax) = \frac{1}{a}\delta(x)$

$$\int_{-\infty}^{+\infty} f(x)\,\delta(ax)\,dx = \frac{1}{a}\int_{-\infty}^{+\infty} f(x)\,\delta(ax)\,d(ax)$$

② $\int_{-\infty}^{+\infty} \delta(x-x_0)\,f(x)\,dx = f(x_0)$

③ $g(x_i) = 0$



$$\delta(g(x)) = \sum_i \frac{\delta(x-x_i)}{|g'(x_i)|}$$

$$\int_{-\infty}^{+\infty} f(x)\,\delta(g(x))\,dx = \sum_i \int_{x_i-\varepsilon}^{x_i+\varepsilon} f(x)\,\delta\big((x-x_i)\,g'(x_i)\big)\,dx$$

④ $\int_{-\infty}^{+\infty} f(x)\,\delta'(x-x_0)\,dx = -\int_{-\infty}^{+\infty} f'(x)\,\delta(x-x_0)\,dx = -f'(x_0)$

⑤ $\iiint f(\vec{r_2})\,\delta^3(\vec{r_2}-\vec{r_1})\,dV = f(\vec{r_1})$

⑥ $\delta(t-x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega(t-x)}\,d\omega$

5.4.

$$\nabla\cdot\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} = 4\pi\,\delta(\vec{r}-\vec{r}')$$

$$\nabla\cdot\frac{\vec{e}_r}{r^2} = \frac{1}{r^2\cancel{\sin\theta}}\left(\frac{\partial}{\partial r}\left(r\cancel{\sin\theta}\,\frac{1}{r^2}\right)+\cdots\right) = 0$$

$$\iiint\left(\nabla\cdot\frac{\vec{e}_r}{r^2}\right)dV = \oiint \frac{\vec{e}_r}{r^2}\cdot d\vec{S} = 4\pi$$

5.5. Green's function.

$$\nabla^2 G(x,x') = \delta(x-x') \quad \Rightarrow G(x,x') = -\frac{1}{4\pi|x-x'|}$$

5.6 Kronecker Delta

$$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$

$$\sum_{ij} f_{ij}\,\delta_{ij} = \sum_i f_{ii}, \qquad f_{ij}\,\delta_{ij} = f_{ii} = f_{jj}$$

$$\sum_j \delta_{ij}\,\delta_{jk} = \delta_{ik}, \qquad \delta_{ij}\,\delta_{jk} = \delta_{ik}$$

5.7. Levi - Civita

$$\varepsilon_{ijk} = \begin{cases} +1 & (1,2,3) \\ -1 & (3,2,1) \\ 0 & \text{else} \end{cases}$$

$$\vec{C} = \vec{A} \times \vec{B} = (A_2 B_3 - A_3 B_2 , \ \cdots \ ) \qquad \vec{A} \cdot \vec{B} = \delta_{ij} A_i B_j$$

$$= \varepsilon_{ijk} A_i B_j \vec{e_k} \qquad\qquad = A_i B_i$$

$$(\vec{A} \times \vec{B}) \cdot \vec{C} = \varepsilon_{ijk} A_i B_j C_k = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$$\varepsilon_{ijk} = \varepsilon_{abc} \delta_{ia} \delta_{jb} \delta_{kc} = \begin{vmatrix} \cancel{\delta_{i1}} & \cancel{\delta_{i2}} & \cancel{\delta_{i3}} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{vmatrix}$$

$$\varepsilon_{lmn} = \begin{vmatrix} \delta_{l1} & \delta_{l2} & \delta_{l3} \\ \delta_{m1} & \delta_{m2} & \delta_{m3} \\ \delta_{n1} & \delta_{n2} & \delta_{n3} \end{vmatrix} = \begin{vmatrix} \delta_{l1} & \delta_{m1} & \delta_{n1} \\ \delta_{l2} & \delta_{m2} & \delta_{n2} \\ \delta_{l3} & \delta_{m3} & \delta_{n3} \end{vmatrix}$$

$$\varepsilon_{ijk} \varepsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix}$$

① $$\varepsilon_{ijk} \varepsilon_{lmk} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{ik} \\ \delta_{jl} & \delta_{jm} & \underline{\delta_{jk}} \\ \delta_{kl} & \delta_{km} & \underline{\delta_{kk}} \end{vmatrix} = 3 (\delta_{il} \delta_{jm} - \delta_{jl} \delta_{im}) -$$

$$\delta_{jk} (\delta_{il} \delta_{km} - \delta_{im} \delta_{kl}) +$$

$$\delta_{ik} (\delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl})$$

$$= 3 (\delta_{il} \delta_{jm} - \delta_{jl} \delta_{im}) - (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) + (\delta_{jl} \delta_{im} - \delta_{jm} \delta_{il})$$

$$= \delta_{il} \delta_{jm} - \delta_{jl} \delta_{im}$$

② $$\varepsilon_{ijk} \varepsilon_{ljk} = \delta_{il} \delta_{jj} - \delta_{jl} \delta_{ij} = 3 \delta_{il} - \delta_{il} = 2 \delta_{il}$$

③ $\varepsilon_{ijk}\varepsilon_{ijk} = 2\delta_{ii} = 6$

习题：

1. (1.1.10) $\sum_{n=1}^{\infty}\left[\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\right]^2$

2. (1.1.12) Catalan's constant.

$$\beta(2) = \sum_{k=0}^{\infty}(-1)^k(2k+1)^{-2} \qquad \text{6-digit}$$

3. (1.5.2 ~ 1.5.3)

$$\alpha_p = \sum_{n=1}^{\infty}\frac{1}{n(n+1)\cdots(n+p)} = \frac{1}{p\,p!}$$

4. (1.5.5) Euler.

5. (1.10.7) $\int_0^x \mathrm{erf}(t)\,dt$