分析与代数基础

一、无穷级数 (infinite series)

$$S_i = \sum_{n=1}^{i} u_n$$

partial sum

部分和序列 $\underline{S = \lim_{i\to\infty} S_i}$，收敛

necessary condition

必要条件：$\lim_{n\to\infty} S_i - S_{i+1} = \lim_{n\to\infty} u_n = 0$.

criterion

Cauchy 收敛准则，$\forall \varepsilon > 0$, $\exists N$, s.t. $|S_j - S_i| < \varepsilon$ $\forall i, j > N$.

发散：S 不存在。

$\begin{cases} S \to \infty \\ \text{振荡的} \end{cases}$

$-1, +1, -1, +1 \cdots$

Example 1.

$1 + r + r^2 + \cdots + \cdots$

$$S_n = \frac{1-r^n}{1-r}$$

if $|r| < 1$，$S = \frac{1}{1-r}$

Example #2

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \cdots$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \cdots > 1 + \frac{1}{2} + \frac{1}{2} + \cdots$$

$\to \infty$

Example 3.

$$\sum_{n} \frac{n}{2^n}$$

$$\frac{a_{n+1}}{a_n} = \cdots \leq \frac{3}{4}$$

(一) 正项级数审敛

Comparison test

1. 比较，$0 \leq u_n \leq a_n$，(用 Cauchy 理解)

$0 \leq b_n \leq u_n$，　n^{-p}，$p<1$ 发散

① Cauchy Root Test　$\lim a_n^{1/n} < 1$　$\lim a_n^{1/n} = 1$　$\lim a_n^{1/n} > 1$

For large n，$(a_n)^{1/n} \leq r < 1$，收敛，$(a_n)^{1/n} \geq 1$ 发散。

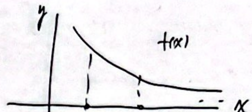
② D'Alembert Ratio Test　同 · 一 · 理

For large n，$a_{n+1}/a_n \leq r < 1$，收敛。$a_{n+1}/a_n \geq 1$ 发散。

保序性　$a_n \leq b_n \Rightarrow \lim a_n \leq \lim b_n$

$\lim a_n < \lim b_n \Rightarrow a_n < b_n$

3. Cauchy Integral Test



$f(x)$ 连续、单调减

$f(n) = a_n$，$S_i = \sum_{n=1}^{i} a_n = \sum_{n=1}^{i} f(n)$

$$S_i \geq \int_1^{i+1} f(x)\,dx$$

$$S_i - a_1 \leq \int_1^{i} f(x)\,dx$$

$$\Rightarrow \underbrace{\int_1^{\infty} f(x)\,dx \leq \sum_{n=1}^{\infty} a_n}_{\text{同敛散}} \leq \int_1^{\infty} f(x)\,dx + a_1$$

估计余项。(Remider)

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n$$

$$\Rightarrow \int_{N+1}^{\infty} f(x)\,dx \leq \sum_{n=N+1}^{\infty} a_n \leq \int_{N+1}^{\infty} f(x)\,dx + a_{N+1}$$

只需 $f(x)$ 连续。

向下取整 $\lfloor x \rfloor$

$$\sum_{n=N_1+1}^{N_2} f(n) = \int_{N_1}^{N_2} f(x)\,dx + \int_{N_1}^{N_2} (x - [x])\, f'(x)\,dx$$

Proof:

$$\begin{cases} \int_{N_1}^{N_2} x f'(x)\,dx = N_2 f(N_2) - N_1 f(N_1) - \int_{N_1}^{N_2} f(x)\,dx \\ \int_{N_1}^{N_2} [x] f'(x)\,dx = \sum_{n=N_1}^{N_2-1} n \int_n^{n+1} f'(x)\,dx = \sum_{n=N_1}^{N_2-1} n\,[f(n+1) - f(n)] \end{cases}$$

$$= -\sum_{n=N_1+1}^{N_2} f(n) - N_1 f(N_1) + N_2 f(N_2)$$

Example 4.

$$\zeta(p) = \sum_{n=1}^{\infty} n^{-p}, \quad \int_1^{\infty} x^{-p}\,dx = \begin{cases} \frac{x^{-p+1}}{-p+1}\Big|_1^{\infty} & p \neq 1 \\ \ln x\Big|_1^{\infty} & p = 1. \end{cases}$$

$$\gamma = \lim_{n\to\infty}\left(\sum_{m=1}^{n} m^{-1} - \ln n\right) \Rightarrow \text{Euler–Mascheroni Constant}.$$

$\ln x$、x^a、e^x.

Example 5.

$$S = \sum_{n=2}^{\infty} \frac{1}{n \ln n} \qquad S = \sum_{n=3}^{\infty} \frac{1}{n \ln(\ln n)} \qquad S = \sum_{n=2}^{\infty} \frac{1}{n \ln^2 n}.$$

$\int$

4. More sensitive Test (正项)

Kummer's theorem,

$\sum_n u_n$ 收敛 if $\lim_{n\to\infty}\left(a_n\frac{u_n}{u_{n+1}}-a_{n+1}\right)\geq C>0$

$\sum_n a_n^{-1}$ 发散 if — — — — — — — ≤ 0 $\Rightarrow \sum_n u_n$ 发散.

若 $\sum_n a_n^{-1}$ 收敛.

$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} > \lim_{n\to\infty}\frac{a_{n+1}}{a_n} > 1$ ✓ 显然..

若 $\sum_n a_n^{-1}$ 发散. 依然成立, 且 $\sum_n a_n^{-1}$ 发散越慢, 判别越强.

$$u_{N+1} \leq (a_N u_N - a_{N+1}u_{N+1})/C$$
$$u_{N+2} \leq (a_{N+1}u_{N+1} - a_{N+2}u_{N+2})/C \quad\Rightarrow\quad \sum_{i=N+1}^{n} u_i \leq \frac{a_N u_N}{C} - \frac{a_n u_n}{C} < \frac{a_N u_N}{C}$$
$$\vdots$$
$$u_n \leq (a_{n-1}u_{n-1} - a_n u_n)/C$$

快 $\to 0$

例. 取 $\cancel{a_n = n,\ \lim\left(n\frac{u_n}{u_{n+1}} - n - 1\right) > 0}$

$\cancel{\lim_n \frac{u_n}{u_{n+1}} > \lim_n \frac{n+1}{n} = 1}$

~~取~~ $\cancel{a_n =}$

D'Alembert 取 $a_n = 1$

Raabe 取 $a_n = n$

Bertrand's 取 $a_n = n\ln n$

— — — — $n\ln n\ln(\ln n)\cdots$ $? = 1$

— — — —

Extended Bertrand's $a_n = n\prod_{k=1}^{K}\ln_{(k)} n$

Gauss' Test. For large n.

$$\frac{u_n}{u_{n+1}} = 1+\frac{h}{n}+\frac{B(n)}{n^2}$$

$B(n)$ is bounded, then if $h>1$, $\sum u_n$ converge, $h\leq 1$, $\sum u_n$ diverge

Confirm, Take $a_n = n\ln n$. Kummer's Theorem

$$\lim_n\left[n\ln n\left(1+\frac{h}{n}+\frac{B(n)}{n^2}\right)-(n+1)\ln(n+1)\right]$$
$$=\lim_n\left[(n+1)\ln n+(h-1)\ln n+\frac{B(n)\ln n}{n}-(n+1)\ln(n+1)\right]$$
$$=\lim_n\left[-(n+1)\ln\left(\frac{n+1}{n}\right)+(h-1)\ln n\right]$$

$h\leq 1$ — —. $h>1$

$$(n+1)\ln\left(\frac{n+1}{n}\right)\sim(n+1)\frac{1}{n}\sim 1$$

Example 6.

$$\frac{a_{2j+2}}{a_{2j}} = \frac{2j(2j+1)-\lambda}{(2j+1)(2j+2)} \Rightarrow \frac{u_j}{u_{j+1}} = \frac{(2j+1)(2j+2)}{2j(j+1)-\lambda}$$

$$\longrightarrow \frac{2j+2}{2j}+\frac{B(j)}{j^2} = 1+\frac{1}{j}+\frac{B(j)}{j^2} \qquad \text{divergent.}$$

D. Alternating series.

1. Leibniz criterion, if a_n monotonically decreasing. $\lim_n a_n = 0$, $\sum_n (-1)^{n+1}a_n$ 收敛.

$$R_{2n} = (a_{2n+1}-a_{2n+2})+(a_{2n+3}-a_{2n+4})+\cdots$$
$$= a_{2n+1}-(a_{2n+2}-a_{2n+3})-(a_{2n+4}-a_{2n+5})$$

$\Rightarrow$ $0<R_{2n}<a_{2n+1}$ error is negative.

Example 7, $S=\sum_{n=1}^{\infty}\frac{\cos(nx)}{n}$.

$$S=\int_1^{\infty}\frac{\cos nx}{n}\,dx+\int_1^{\infty}(n-[n])\left(-\frac{x}{n}\sin(nx)-\frac{\cos nx}{n^2}\right)dn$$
$$=\left[\frac{\sin nx}{nx}\right]_1^{\infty}+\frac{1}{x}\int_1^{\infty}\frac{\sin nx}{n^2}\,dn + \cdots + \int_1^{\infty}(n-[n])\frac{\sin nx}{n}\,dn$$

Let $(n-[n])\sin(nx) = g'(n)$ $\quad g(N)=\int_1^N (n-[n])\sin nx\,dx$.

$$=\int_1^{\infty}\frac{g'(n)}{n}\,dn = \left[\frac{g(n)}{n}\right]_1^{\infty}+\int_1^{\infty}\frac{g(n)}{n^2}\,dn \quad ✓.$$

Dirichlet ~~Abel~~

$$\sum_{n=1}^{\infty}\frac{\cos(nx)}{n} = \mathrm{Re}\left[\sum_{n=1}^{\infty}\frac{e^{inx}}{n}\right] = \mathrm{Re}\left[-\ln(1-e^{ix})\right]$$

Because of

$$x+\frac{1}{2}x^2+\frac{1}{3}x^3+\cdots = f(x)$$

$$f'(x) = 1+x+x^2+x^3+\cdots = \frac{1}{1-x}$$

$$f(x) = -\ln(1-x)$$

取共轭 $\overline{f(i)} = f(-i)$

$$\left[-\ln(1-e^{ix}) - \ln(1-e^{-ix})\right]\frac{1}{2}$$

$$= -\frac{1}{2}\ln\left(1+1-e^{ix}-e^{-ix}\right)$$

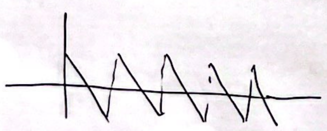
$$= -\frac{1}{2}\ln\left(2-2\cos x\right) = -\ln\left(2\sin\frac{x}{2}\right)$$

$$\sum_{n=1}^{\infty}\frac{\sin nx}{n} = \mathrm{Im}\left[\sum_{n=1}^{\infty}\frac{e^{inx}}{n}\right] = \mathrm{Im}\left[-\ln(1-e^{ix})\right]$$

$$\left[-\ln(1-e^{ix}) + \ln(1-e^{-ix})\right]\frac{1}{2i}$$

$$= \frac{1}{2i}\ln\left(\frac{1-e^{-ix}}{1-e^{ix}}\right) = \frac{1}{2i}\ln\left(-e^{-ix}\right) = \frac{1}{2i}\ln\left(e^{i(\pi-x+2k\pi)}\right)$$

$$\cancel{= \frac{1}{2i}\ln\frac{}{2-2\cos\theta}} = \frac{1}{2}(\pi-x+2k\pi)$$



$\sum_n a_n b_n$

Abel $\{b_n\}$ 单有界, $\sum_{n=1}^{\infty} a_n$ 收敛 ✓

Dirichlet $\{b_n\} \xrightarrow{单} 0$, $\sum_{n=1}^{\infty} a_n$ 部分有界 ✓

~~Abs~~ Absolute / Conditional Convergence

$S = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ Condition Convergence.

$S = \ln 2$

$$S = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\cdots$$

$$\frac{1}{2}S = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots$$

$$\Rightarrow \frac{3}{2}S = 1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}\cdots = \frac{3}{2}\ln 2$$

Riemann's theorem.

Absolutely convergent ✓

Example 8.

$$\left[\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}\right]^2 = \sum_n (-1)^n\left[\frac{1}{\sqrt{1}}\frac{1}{\sqrt{n-1}}+\frac{1}{\sqrt{2}}\frac{1}{\sqrt{n-2}}+\cdots+\frac{1}{\sqrt{n-1}}\frac{1}{\sqrt{1}}\right].$$

$\not\to 0$

Improvement of Convergence.

$$\alpha_1 = \sum_{n=1}^{\infty}\frac{1}{n(n+1)} = 1$$

$$\alpha_2 = \sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)} = \frac{1}{4}$$

$$\alpha_3 = \sum_{n=1}^{\infty}\frac{1}{n\cdots(n+3)} = \frac{1}{18},$$

$$\alpha_p = \sum_{n=1}^{\infty}\frac{1}{n\cdots(n+p)} = \frac{1}{p\,p!}.$$

Proof:

$$\frac{1}{n(n+1)(n+2)} = \frac{A}{n}+\frac{B}{n+1}+\frac{C}{n+2}$$

$$= \frac{A(n+1)(n+2)+Bn(n+2)+C(n+1)n}{n(n+1)(n+2)}$$

$$= \frac{(A+B+C)n^2+(3A+2B+C)n+2A}{}$$

$\Rightarrow A = 1/2$, $\begin{cases}\frac{1}{2}+B+C=0\\ \frac{3}{2}+2B+C=0\end{cases} \Rightarrow B=-1,\ C=\frac{1}{2}$

$$\frac{\frac{1}{2}}{n-1} \quad \frac{-1}{n} \quad \frac{\frac{1}{2}}{n+1}$$

$$\Rightarrow \frac{\frac{1}{2}}{n}+\frac{-1}{n+1}+\frac{\frac{1}{2}}{n+2}$$

$$\frac{\frac{1}{2}}{n+1}+\frac{-1}{n+2}+\frac{\frac{1}{2}}{n+3}$$

$\zeta(3) = \sum_n \frac{1}{n^3}$

$\zeta(3) + a\alpha_2$

$= \sum_n \left[\frac{1}{n^3} + \frac{a}{n(n+1)(n+2)} \right] = \sum_n \frac{n^2(1+a)+3n+2}{n^3(n+1)(n+2)}$

Let $a = -1$

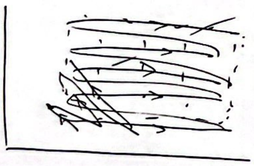
$\Rightarrow \zeta(3) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{3n+2}{n^3(n+1)(n+2)}$.

Example 9.

$\sum_n \frac{1}{1+n^2} = n^{-2}(n^{-2}+1)^{-1} = n^{-2}\left(1 - n^{-2} + n^{-4} - \frac{n^{-6}}{1+n^{-2}}\right)$

$= \zeta(2) - \zeta(4) + \zeta(6) - \sum_{n=1}^{\infty} \frac{1}{n^8+n^6}$

Rearrangement of Double series.



$S = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{n,m}$

$= \sum_{p=0}^{\infty} \sum_{q=0}^{p} a_{p-q,q}$

$= \sum_{r=0}^{\infty} \sum_{s=0}^{[r/2]} a_{s,r-2s}$.

Series of functions.

$\sum_{n=1}^{\infty} u_n(x) = S(x) = \lim_{n\to\infty} S_n(x)$

Uniform Convergence

$\forall \varepsilon > 0, \exists N,$ 与x无关 $\forall n \geq N$ $\forall x \in [a,b]$, s.t. $|S(x) - S_n(x)| < \varepsilon$.

Reminder

~~Properties~~ Example 10, $[0,1]$ $\forall x$, AC.

$S(x) = \sum_{n=0}^{\infty} (1-x)x^n = \begin{cases} 1 & 0 \leq x < 1 \\ 0 & x = 1 \end{cases}$

$\forall x \neq 0$, $S_N = 1 - x^N$

$|1-(1-x^N)| = x^N < \varepsilon$

AC and UC is independent.

Weierstrass M Test (优势级数)

若 $u_i(x) \leq M_i$ $\forall x \in [a,b]$

$\sum M_i$ 收敛 $\Rightarrow \sum_i u_i(x)$ 一致收敛.

Proof: $\sum_{i=n+1}^{\infty} u_i(x) \leq \sum_{i=n+1}^{\infty} M_i < \varepsilon$ ...

$|S(x) - S_n(x)| < \varepsilon$.

Example 11.

$S(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n+x^2}$

$|S(x) - S_n(x)| < |u_{n+1}(x)| \leq |u_{n+1}(0)|$

Abel

Dirichlet.

$x \in [a,b]$

Properties. $\sum_n u_n$ UC, $u_n(x)$ continous.

1. $S(x) = \sum_n u_n(x)$ is C.

2. $\int_a^b S(x)\,dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)\,dx$

3. $\frac{d}{dx} S(x) = \sum_{n=1}^{\infty} \frac{d}{dx} u_n(x)$

$\left(\begin{array}{l} \frac{du_n(x)}{dx} \text{ is C on } [a,b] \\ \sum \frac{d}{dx} u_n(x) \text{ is UC on } [a,b] \end{array} \right)$

Taylor Series.

$f(x+h) = \cancel{f(x)} \sum_{n=0}^{\infty} \frac{h^n}{n!} f^{(n)}(x) = \sum_{n=0}^{\infty} \frac{h^n D^n}{n!} f(x) = e^{hD} f(x)$

Radius of convergence

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = R^{-1}.$$

For $0<S<R$, $\forall x\in[-S,S]$, it is A an U C. (Wierostrass M)

Uniqueness Theorem.

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad -R_a < x < R_a$$

$$= \sum_{n=0}^{\infty} b_n x^n, \quad -R_b < x < R_b.$$

$\Rightarrow \quad a_n = b_n.$

Binomial Theorem

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2}x^2 + \cdots + R_n$$

$$R_n = \frac{x^n}{n!}(1+\xi)^{m-n}\, m(m-1)\cdots(m-n+1)$$

$-1<x<1$

$$\binom{m}{n} = \frac{m(m-1)\cdots(m-n+1)}{n!} = \frac{m!}{n!(n-m)!}$$

$$(1+x)^m = \sum_{n=0}^{\infty}\binom{m}{n}x^n$$

For $m<0$, $m=-p$.

$$\binom{-p}{n} = \frac{(-p)(-p-1)\cdots(-p-n+1)}{n!} = (-1)^n \frac{p(p+1)\cdots(p+n-1)}{n!} = (-1)^n\frac{(p+n-1)!}{n!(p-1)!}$$

Pochhammer Symbol

$(a)_n$, $(a)_0 = 1$, $(a)_1 = a$, $(a)_{n+1} = a(a+1)\cdots(a+n)$

$$\binom{m}{n} = \frac{(m-n+1)_n}{n!}$$

$$\binom{-\frac{1}{2}}{n} = \frac{1}{n!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{\cancel{2n+1}\ 2n-1}{2}\right) = (-1)^n\frac{1\cdot 3\cdots(2n-1)}{2^n\cdot n!} = (-1)^n\frac{(2n-1)!!}{(2n)!!}$$

$0!! = (-1)!! = 1$

$$(a_1+\cdots+a_m)^n = \sum_{n_1+\cdots+n_m=n}\frac{n!}{n_1!\,n_2!\cdots n_m!}a_1^{n_1}a_2^{n_2}\cdots a_m^{n_m}$$

Mathematical Induction.

一.

二.

Example 12.

$y = \sin^{-1}x$ Taylor.

~~$\sin^{-1}x = y$~~ ~~$y=$~~ $\quad x = \sin y$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \Rightarrow \sin^{-1}x = \int_0^x \frac{dt}{\sqrt{1-t^2}}$$

$$= \int_0^x \sum_{n=0}^{\infty}(-1)^n\frac{(2n-1)!!}{(2n)!!}(-t^2)^n\,dt$$

$$= \sum_{n=0}^{\infty}\cancel{(-1)^n}\frac{(2n-1)!!}{(2n)!!}\frac{x^{2n+1}}{2n+1}.$$

Improving the convergence

$$\ln(1+x) \sim \frac{1}{n}$$

$$(1+a_1x)\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} + a_1\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{n+1}}{n}$$

$$= x + \sum_{n=2}^{+\infty}(-1)^{n-1}\left(\frac{1}{n} - \frac{a_1}{n-1}\right)x^n$$

$$= x + \quad \cdots$$

Let $a_1 = 1$,

$$\ln(1+x) = \left(\frac{x}{1+x}\right)\left(1 - \sum_{n=1}^{\infty}\frac{(-1)^n}{n(n+1)}x^n\right)$$

Partial fraction expansions [illegible]

Improve the range, Euler.

Vectors and Complex numbers.

1. Functions in the complex domain

$$e^{z} = 1 + z + \frac{1}{2!}z^2 + \cdots$$

$$e^{iz} = \cos z + i\sin z.$$

2. Polar representation.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow x + iy = re^{i\theta}$$

$$\overline{f(i)} = f(-i)$$

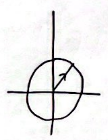
$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

$$\big||z_1| - |z_2|\big| \le |z_1 + z_2| \le |z_1| + |z_2|$$

$$\operatorname{Re} z = \frac{z+\bar{z}}{2} \qquad \operatorname{Im}(z) = \frac{z-\bar{z}}{2i}$$

3. $|z| = 1$,

$$z = e^{i\theta}$$



4. Circular and hyperbolic function.

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

$$\cosh\theta = \frac{e^{\theta}+e^{-\theta}}{2} \qquad \sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2}$$

$$\cosh iz = \cos z \qquad \sinh iz = i\sin z$$

$$\cos iz = \cosh z \qquad \sin iz = i\sinh z.$$

$$e^{i\theta} = i\sin\theta + \sqrt{1-\sin^2\theta} = i\frac{\tan\theta}{\sqrt{1+\tan^2\theta}} + \frac{1}{\sqrt{1+\tan^2\theta}}$$

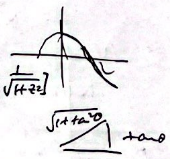
$$\sin^{-1} z = -i\ln\left[iz + \sqrt{1-z^2}\right]$$

$$\tan^{-1} z = -i\ln\left[i\frac{z}{\sqrt{1+z^2}} + \frac{1}{\sqrt{1+z^2}}\right]$$

$$= -i\ln\left[\frac{1+zi}{\sqrt{(1+zi)(1-zi)}}\right]$$

$$= -i\ln\left[\sqrt{\frac{1+zi}{1-zi}}\right]$$

$$= i/2\left(\ln[1-iz] - \ln(1+zi)\right)$$



$$\sinh^{-1} z = \ln\left[z + \sqrt{1+z^2}\right]$$

$$\tanh^{-1} z = \frac{1}{2}\left[\ln(1+z) - \ln(1-z)\right]$$

5. Power and Roots.

$a \in \mathbb{C}$.

$z = re^{i\varphi}$, $z^n = r^n e^{in\varphi}$, 怎么 z^a ?

$z = re^{i\varphi}$, $z^{1/n} = r^{1/n} e^{i(\varphi + 2m\pi)/n}$ n-valued.

例

$$1^{1/3} = 1,\ e^{i\frac{2\pi}{3}},\ e^{i\frac{4\pi}{3}}.$$

$$z^a = e^{a\ln z}$$

6. Logarithm

$\cancel{\ln z} = \ln r + i\theta$

$\ln z = \ln r + i(\theta + 2n\pi)$.

注意!

$\cancel{z^a = e^{a\ln z}}$

$\cancel{z^a = \ln e^{z^a}}$ ✗

$x^y = e^{y\ln x}$ ✓

$\cancel{\ln w^z = z\ln w \ (\text{mod } 2\pi i)}$

$x = \ln e^x$ ✗ $\rightarrow$ $\ln(-1)^2 \ne 2\ln(-1)$

$\ln(ab) = \ln a + \ln b$ ✗ $\rightarrow$ $\cancel{\log}\ \ln(-i) = \ln 1 - \frac{i\pi}{2} = -\frac{i\pi}{2}$

$\ne$ $\ln(-1) + \ln(i) = \frac{3\pi i}{2}$.

$\cancel{(e^a)^b}$ $(a^b)^c \ne (a^c)^b \ne a^{bc}$

$\rightarrow ((-1)^2)^{1/2} \ne ((-1)^{1/2})^2 \cdots$

Fail of power and logarithm identity

1. $\cancel{\log w^z}$ $\cancel{w^z = w\, e^{(\ln w^z + i2\pi n)}}$ $e^{z\ln w} = e^{z[\operatorname{Ln} w + i2\pi n]}$ $\cancel{= \operatorname{Ln} w}$ ✓

$\cancel{\ln w^z = \ln w^z + i2\pi n = z\ln w} + z$

$\{\cancel{\log_w z}\ \ln w^z\} = \{z\operatorname{Ln} w + z\cdot 2\pi i n + 2\pi i m\}$

$\{z\ln w\} = \{z\operatorname{Ln} w + z\cdot 2\pi i n\}$

2. $[(-1)\cdot(-1)]^{1/2} = \{1, -1\}$.

Derivatives and extrema.

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

1. $$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds}$$

2. $$\left(\frac{\partial y}{\partial x}\right)_f = - \frac{\left(\frac{\partial f}{\partial x}\right)_y}{\left(\frac{\partial f}{\partial y}\right)_x}$$

$\frac{df}{ds} = 0,\ \forall \vec{s} \Leftrightarrow \frac{\partial f}{\partial x_i} = 0$ stationary point

$f''_{xx} f''_{yy} - f''_{xy} > 0$ ✓ $A < 0$ 极大

< 0 ✗

$= 0$ 不确定

Evaluation of integrals.

1. Integration by parts.
2. Special functions

Gamma $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$

Factorial $n! = \int_0^\infty t^n e^{-t} dt$

Riemann Zeta $\zeta(x) = \frac{1}{\Gamma(x)} \int_0^{+\infty} \frac{t^{x-1} dt}{e^t - 1}$

exponential Integral $E_n(x) = \int_1^\infty t^{-n} e^{-t} dt$

Sine —— $Si(x) = -\int_x^{+\infty} \frac{\sin t}{t} dt$

Cosine —— $Ci(x) = -\int_x^\infty \frac{\cos t}{t} dt$

Error function $erf = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ $erf(\infty) = 1$

$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^{+\infty} e^{-t^2} dt$ $erfc(x) = 1 - erf(x)$

Dilogarithm $Li_2(x) = -\int_0^x \frac{\ln(1-t)}{t} dt$

Example

1. $I = \int_0^\infty \frac{e^{-x^2}}{x^2+a^2} dx$

$$J(t) = \int_0^{+\infty} \frac{e^{-t(x^2+a^2)}}{x^2+a^2} dx$$

$$I = e^{a^2} J(1)$$

$$J'(t) = -\int_0^\infty e^{-t(x^2+a^2)} dx = -e^{-ta^2} \int_0^{+\infty} e^{-tx^2} dx = -\frac{1}{2}\sqrt{\frac{\pi}{t}}\, e^{-ta^2}$$

$$J(t) = \frac{\sqrt{\pi}}{2} \int_t^{+\infty} \frac{e^{-t'a^2}}{t'^{1/2}} dt' = \frac{\sqrt{\pi}}{a} \int_{at^{1/2}}^\infty e^{-u^2} du$$

$$\Rightarrow I = \frac{\pi}{2a} e^{a^2} erfc(a)$$

2. $I = \int_0^1 \ln\left(\frac{1+x}{1-x}\right) \frac{dx}{x}$

$$= \int_0^1 dx \cdot 2\left[1 + \frac{x^2}{3} + \frac{x^4}{5} + \cdots\right] = 2\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\frac{1}{2^2}\zeta(2) = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} \cdots$$

$$I = \frac{3}{2}\zeta(2)$$

$dx_1 dx_2 \cdots dx_n \Rightarrow dy_1 \cdots dy_n$

$dx_1 \cdots dx_n = J\, dy_1 \cdots dy_n$

$$J = \frac{\partial(x_1 \cdots x_n)}{\partial(y_1 \cdots y_n)}$$

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3. $I = \int_0^\infty e^{-at} \cos bt\, dt$

$$= Re \int_0^{+\infty} e^{(a+ib)t} dt = Re\left(\frac{a+ib}{a^2+b^2}\right)$$

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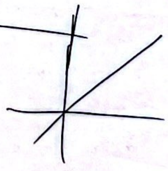
4. $\int_0^{+\infty} e^{-r} dr \int_r^\infty \frac{e^{-s}}{s} ds$



$$= \int_0^\infty \frac{e^{-s}}{s} ds \int_0^s e^{-r} dr$$

$$= \int_0^\infty e^{-s}/s\, (1 - e^{-s}) ds$$

$$= \int_0^\infty \frac{e^{-s}}{s} \sum_{n=1}^\infty \frac{(-1)^{n-1} s^n}{n!} ds = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n!} \int_0^\infty s^{n-1} e^{-s} ds = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n!} (n-1)! = \ln 2$$

Dirac Delta function

$$\delta(x) = 0 \qquad x \neq 0$$

$$\begin{cases} f(0) = \int_a^b f(x)\,\delta(x)\,dx \end{cases} \Rightarrow \int_{-\infty}^{+\infty} \delta(x)\,dx = 1$$

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1. $$\delta_n = \begin{cases} 0 & x < -\frac{1}{2n} \\ n & -\frac{1}{2n} < x < \frac{1}{2n} \\ 0 & x > \frac{1}{2n} \end{cases}$$

$$\delta_n = \frac{n}{\sqrt{\pi}} \exp(-n^2 x^2) \qquad \left[\int_{-\infty}^{+\infty} e^{-n^2x^2} dx = \sqrt{\frac{\pi}{n}} \right]$$

$$\delta_n = \frac{n}{\pi} \frac{1}{1+n^2x^2}$$

$$\delta_n = \frac{\sin nx}{\pi x} = \frac{1}{2\pi} \int_{-n}^{n} e^{ixt}\,dt \Rightarrow \delta(x) = \lim_{n} \delta_n(x)$$

2. linear functional.

Property of $\delta(x)$

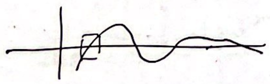
1. $$\delta(ax) = \frac{1}{a}\,\delta(x)$$

$$\int_{-\infty}^{+\infty} f(x)\,\delta(ax)\,dx \quad \cdots$$

2. $$\int_{-\infty}^{+\infty} \delta(x - x_0)\,f(x)\,dx = f(x_0)$$ [illegible]

3. $$g(x_i) = 0$$

$$\delta(g(x)) = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|}$$



$$\int_{-\infty}^{+\infty} f(x)\,\delta(g(x))\,dx = \sum_i \int_{x_i-\varepsilon}^{x_i+\varepsilon} f(x)\,\delta\Big((x - x_i)\,g'(x_i)\Big)\,dx$$ ✓

4. $$\int_{-\infty}^{+\infty} f(x)\,\delta'(x - x_0)\,dx = -\int_{-\infty}^{+\infty} f'(x)\,\delta(x - x_0)\,dx = -f'(x_0)$$

5. $$\iiint f(\vec{r}_2)\,\delta^3(\vec{r}_2 - \vec{r}_1)\,dV = f(\vec{r}_1)$$

6. $$\delta(t - x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega(t-x)}\,d\omega.$$

$$\nabla \cdot \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} = 4\pi\,\delta(\vec{r} - \vec{r}')$$

$$= \nabla \cdot \frac{\vec{e}_r}{r^2}$$

$$= \frac{1}{r^2 \sin\theta} \left[\frac{\partial}{\partial r}(\sin\theta) \right) + \cdots + \cdots = 0$$

Green's Function

~~Linear~~

~~Differential operator~~ $\cancel{\nabla^2 G =}\ \delta$

$$\nabla^2 G(x, x') = \delta(x - x') \Rightarrow G(x, x') = -\frac{1}{4\pi |x - x'|}$$

potential.

Kronecker Delta

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$\sum_{ij} f_{ij}\,\delta_{ij} = \sum_i f_{ii}$$

$$f_{ij}\,\delta_{ij} = f_{ii} = f_{jj}$$

$$\delta_{ij}\,\delta_{jl} = \delta_{il}$$

Levi-Civita.

$$\varepsilon_{ijk} = \begin{cases} +1 & (1,2,3) \\ -1 & (3,2,1) \\ 0 & \text{else} \end{cases}$$

$\cancel{\delta_{ij}}$ $\cancel{\varepsilon_{ijk} = \varepsilon_{lmn}\,\delta_{mj}\,\delta_{nk}}$

$$\begin{vmatrix} \cancel{e_1} & \cancel{e_2} & \cancel{e_3} \\ \cancel{\delta_{1j}} & \cancel{\delta_{2j}} & \cancel{\delta_{3j}} \\ \delta_{1k} & \delta_{2k} & \delta_{3k} \end{vmatrix}$$

$$\varepsilon_{ijk}\,\varepsilon_{lmn} = \begin{vmatrix} & & \\ & & \end{vmatrix}$$

$\cancel{\varepsilon_{ijk}\,\varepsilon_{lmn}}$

$= \cancel{\varepsilon_{lmn}\ \varepsilon_{abk}\ \delta_{ai}\,\delta_{bj}}\ \varepsilon_{cdn}\ \delta_{cl}\,\delta_{dm}$

$$\varepsilon_{ijk} = \varepsilon_{abc}\,\delta_{ia}\,\delta_{jb}\,\delta_{kc}$$

$$= \begin{vmatrix} \delta_{1a} & \delta_{2a} & \delta_{3a} \\ \delta_{1b} & \delta_{2b} & \delta_{3b} \\ \delta_{1c} & \delta_{2c} & \delta_{3c} \end{vmatrix}$$

$\cdots$

习题：

1. $\sum_{n=1}^{\infty}\left[\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\right]^2$ (1.1.10)

2. Catalan's constant

$\beta(2)=\sum_{k=0}^{\infty}(-1)^k(2k+1)^{-2}$ 6-digit (1.1.12)

3. $\alpha_p=\sum_{n=1}^{\infty}\frac{1}{n(n+1)\cdots(n+p)}=\frac{1}{p\,p!}$ (1.5.3)

(1.5.2)

4. 1.5.5 Euler transformation.

5. $\int_0^x \mathrm{erf}(t)\,dt$. (1.10.7)