代数基础选讲

1. Determinants

1.1. Homogeneous Linear equation

Inhomogeneous Linear equation

Crammer's rule

1.2. Levi - Civita

$$\varepsilon_{ij\cdots} = \begin{cases} +1 & ij\cdots \text{ is an even permutation} \\ -1 & ij\cdots \text{ is an odd permutation} \\ 0 & \text{else} \end{cases}$$

$$D_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ - & - & - & - \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{ij\cdots} \varepsilon_{ij\cdots} a_{1i} a_{2j} \cdots$$

$$\varepsilon_{i_1 i_2 \cdots i_n} = \varepsilon_{j_1 j_2 \cdots j_n} \delta_{i_1 j_1} \delta_{i_2 j_2} \cdots \delta_{i_n j_n} = \begin{vmatrix} \delta_{i_1 1} & \delta_{i_2 1} & \cdots & \delta_{i_n 1} \\ - & - & - & - \\ \delta_{i_1 n} & \delta_{i_2 n} & \cdots & \delta_{i_n n} \end{vmatrix}$$

1.3 Properties.

1.4. Minors

$$D_n = \sum_{j=1}^{n} a_{ij} \underbrace{(-1)^{i+j} M_{ij}}_{\text{co factor}}$$

$\rightarrow$ Minors

2. Matrices

2.1. $A+B$. $\underline{AB}$. αA

2.2. $AB \neq BA$, For squares, we can define commutator 对易子

$$[A, B] = AB - BA$$

Poisson Bracket

$$\{u, v\} = \sum_i \left(\frac{\partial u}{\partial q_i} \frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i} \frac{\partial v}{\partial q_i} \right)$$

① $\{u, v\} = -\{v, u\}$

② $\{u, c\} = 0$

③ $\{u_1 + u_2, v\} = \{u_1, v\} + \{u_2, v\}$

④ $\{u, v_1 + v_2\} = \{u, v_1\} + \{u, v_2\}$

⑤ $\{u_1 u_2, v\} = \{u_1, v\} u_2 + u_1 \{u_2, v\}$

⑥ $\{u, v_1 v_2\} = \{u, v_1\} v_2 + v_1 \{u, v_2\}$

⑦

$$\{u, \{v, w\}\} + \{v, \{w, u\}\} + \{w, \{u, v\}\} = 0$$

$$[u, v] = D\{u, v\}$$

$$[\hat{x}, \hat{p}_x] = \hat{x}\hat{p}_x - \hat{p}_x \hat{x} = D\left\{ \frac{\partial x}{\partial x} \frac{\partial p_x}{\partial p_x} - \frac{\partial x}{\partial p_x} \frac{\partial p_x}{\partial x} + \cdots \right\} = D = i\hbar$$

$$(AB)C = A(BC)$$

Example 1. Pauli Matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_1 \sigma_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \sigma_2 \sigma_1 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$[\sigma_1, \sigma_2] = 2i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2i\sigma_3.$$

Example 2.

$$A = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad B = \begin{pmatrix} 4 & 5 & 6 \end{pmatrix}$$

$$AB = \begin{pmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{pmatrix} \qquad BA = 4 + 10 + 18 = 32 = \mathrm{tr}(AB)$$

$$(AB)^n = ABABA\cdots B = tr(AB)^{n-1} AB.$$

3. Unit Matrix, $\mathbb{1}$, Diagonal Matrices.

3.1 Matrix Inverse, $|A| \neq 0$, $\exists ! B \in \mathbb{R}^{n\times n}$

$$AB = BA = \mathbb{1}, \quad A^{-1} := B.$$

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \quad AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Matrix is a ring, a nonzero commutative ring in which every nonzero element has a multiplicate inverse is a field.

$$(A^{-1})_{ij} = \frac{(-1)^{i+j} M_{ji}}{\det(A)}, \quad \left((-1)^{i+j} M_{ji}\right) = A^*$$

Adjugate Matrix

Example2. Gauss-Jordon matrix inversion

$$\left(\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & B \\ 0 & 0 & 1 & \end{array}\right). \quad B = A^{-1}$$

3.2. Derivative of determinant.

$$\frac{\partial \det(A)}{\partial a_{ij}} = (-1)^{i+j} M_{ij} = (A^{-1})_{ji} \det(A)$$

$$\frac{d \det(A)}{dx} = \det(A) \sum_{ij} (A^{-1})_{ji} \frac{da_{ij}}{dx}$$

3.3. $\det(AB) = \det(A)\det(B)$

3.4 rank of matrix.

3.5.

3.5.1 Transpose $(\tilde{A})_{ij} = a_{ji}$

$\tilde{A} = A$, symmetric

$\tilde{A} = -A$, Anti-symmetric

3.5.2. complex conjugate $\bar{A} = (\overline{a_{ij}})$

3.5.3. adjoint of matrix (伴随，共轭转置)

$(A^{\dagger})_{ij} = \overline{a_{ji}}$

3.5.4. trace $tr(A) = \sum_{i=1}^{n} a_{ii}$

$tr(A+B) = tr(A) + tr(B)$, $tr(\alpha A + \beta B) = \alpha tr(A) + \beta tr(B)$

$tr(AB) = (AB)_{ii} = a_{ij} b_{ji} = b_{ji} a_{ij} = (BA)_{jj} = tr(BA)$

$tr([A,B]) = 0$

$tr(ABC) = tr(BCA) = tr(CAB)$

trace is a linear operator

determinant is not.

3.5.5. $(AB)^T = B^T A^T$

$(AB)^{\dagger} = B^{\dagger} A^{\dagger}$

$(AB)^{-1} = B^{-1} A^{-1}$

3.5.6. orthogonal Matrices

$S^{-1} = S^T$ or $S^T S = \mathbb{1}$

$\det(S) = \pm 1$

$\det(S) = 1$，转动，$\det(S) = -1$，镜面.转动

S_1, S，$S_1 S_2$ ✓

3.5.7. Unitary Matrices (酉、幺正矩阵)

$U^\dagger = U^{-1}$，$U^\dagger U = U U^\dagger = \mathbb{1}$

$\det(U)\det(U^\dagger) = |\det(U)|^2 = 1$

$\Rightarrow \det(U) = e^{i\theta}$，$\det(U^\dagger) = e^{-i\theta}$

U_1, U_2，$U_1 U_2$ ✓

3.5.8. Hermitian Matrices (Self-Ajoint)

(自伴，厄米，埃尔米特)

$H^\dagger = H$，$\overline{h_{ji}} = h_{ij}$

A, B，AB ×　$AB+BA$，✓　$AB-BA$ anti-Hermitian

$(AB-BA)^\dagger = (AB)^\dagger - (BA)^\dagger = -(AB-BA)$

3.5.9. Direct Product.

$A_{m\times n}$，$B_{m'\times n'}$

$C = A \otimes B$，$C_{mm' \times nn'}$

$C_{\alpha\beta} = A_{ij} B_{kl}$

$$\begin{cases} \alpha = m'(i-1)+k \\ \beta = n'(j-1)+l \end{cases}$$

Example 3.

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$\Rightarrow A \otimes B = \left(\begin{array}{c|c} a_{11} B & a_{12} B \\ \hline a_{21} B & a_{22} B \end{array}\right) = \left(\begin{array}{cc|c} a_{11} b_{11} & a_{11} b_{12} & \\ a_{11} b_{21} & a_{11} b_{21} & \\ \hline & & \cdots \end{array}\right)$$

3.5.10. properties.

$$C = A \otimes B, \quad C' = A' \otimes B'$$

$$CC' = (AA') \otimes (BB')$$

$$C \otimes (A+B) = C \otimes A + C \otimes B$$

$$(A+B) \otimes C = A \otimes C + B \otimes C$$

Example 4. Dirac equation

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_i^2 = \mathbb{1}_2, \quad \sigma_i \sigma_j + \sigma_j \sigma_i = 0 \quad \forall i \neq j.$$

$$E^2 = |p|^2 c^2 + m^2 c^4, \quad \sigma = \sigma_1 \hat{e}_1 + \sigma_2 \hat{e}_2 + \sigma_3 \hat{e}_3$$

$$(\vec{\sigma} \cdot \vec{p}) = p^2 \mathbb{1}_2$$

$$\Rightarrow \quad E^2 \mathbb{1}_2 - c^2 (\vec{\sigma} \cdot \vec{p})^2 = m^2 c^4 \mathbb{1}_2$$

$$(E \mathbb{1}_2 + c \vec{\sigma} \cdot \vec{p})(E \mathbb{1}_2 - c \vec{\sigma} \cdot \vec{p}) \psi_1 = m^2 c^4 \psi_1$$

Define $(E \mathbb{1}_2 - c \vec{\sigma} \cdot \vec{p}) \psi_1 = mc^2 \psi_2$

$$\Rightarrow \quad (E \mathbb{1}_2 + c \vec{\sigma} \cdot \vec{p}) \psi_2 = mc^2 \psi_1$$

$$\Leftarrow \quad \psi_1 = \psi_A + \psi_B, \quad \psi_2 = \psi_A - \psi_B$$

$$\Rightarrow \begin{cases} E\psi_A - c(\vec{\sigma}\cdot\vec{p})\,\psi_B = mc^2\psi_A \\ c\,\vec{\sigma}\cdot\vec{p}\,\psi_A - E\psi_B = mc^2\psi_B \end{cases}$$

$$\left[\begin{pmatrix} E\mathbb{1}_2 & 0 \\ 0 & -E\mathbb{1}_2 \end{pmatrix} - \begin{pmatrix} 0 & c(\vec{\sigma}\cdot\vec{p}) \\ -c(\vec{\sigma}\cdot\vec{p}) & 0 \end{pmatrix}\right]\begin{pmatrix}\psi_A \\ \psi_B\end{pmatrix} = mc^2\begin{pmatrix}\psi_A \\ \psi_B\end{pmatrix}$$

$$\Rightarrow \left[(\sigma_3 \otimes \mathbb{1}_2)E - \gamma \otimes c(\vec{\sigma}\cdot\vec{p})\right]\Psi = mc^2\Psi$$

in which $\Psi = \begin{pmatrix}\psi_A \\ \psi_B\end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}$, $\gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$\gamma_0 = \sigma_3 \otimes \mathbb{1}_2 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$\gamma^i = \gamma \otimes \sigma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

$$\Rightarrow \left[\gamma_0 E - c(\gamma^1 p_1 + \gamma^2 p_2 + \gamma^3 p_3)\right]\Psi = mc^2\Psi$$

令 $\gamma^0\gamma^i = \alpha_i$, $\gamma_0^2 = \mathbb{1}_4$

$$\left[\gamma_0 mc^2 + c(\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3)\right]\Psi = E\Psi$$

Dirac equation

$(\gamma^0)^2 = \mathbb{1}_4$, $(\gamma^i)^2 = -\mathbb{1}_4$ Clifford Algebra

$\gamma^\mu\gamma^i + \gamma^i\gamma^\mu = 0$

Complete Basis

$\mathbb{1}_4$, $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, γ^μ ($\mu = 0 \sim 3$)

$\gamma^5\gamma^\mu$, $\sigma^{\mu\nu} = i\gamma^\mu\gamma^\nu$

3.6. Functions of Matrices.

$$e^A = \sum_{j=0}^{\infty} \frac{1}{j!} A^j$$

$$\sin(A) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} A^{2j+1}$$

$$\cos(A) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} A^{2j}$$

$$\sigma_i^{2k} = \mathbb{1}$$

$$\sigma_i^{2k+1} = \sigma_i$$

$$e^{i\sigma_k \theta} = \cos(\sigma_k \theta) + i\sin(\sigma_k \theta)$$

$$= \mathbb{1}_2 \cos\theta + i\sigma_k \sin\theta$$

$$e^{i\sigma^{\mu\nu}\theta} = \cos(\sigma^{\mu\nu}\theta) + i\sin(\sigma^{\mu\nu}\theta)$$

$$= \mathbb{1}_4 \cos\theta + i\sigma^{\mu\nu} \sin\theta$$

$$e^{i\sigma^{0k}\varphi} = \mathbb{1}_4 \cosh\varphi + i\sigma^{0k} \sinh\varphi$$

Hermitian and Unitary matrices.

U is unitary. $U = e^{iH}$, H is Hermitian

$$U^\dagger = e^{-iH^\dagger} = e^{-iH} = (e^{iH})^{-1} = U^{-1}$$

Trace formula for Hermitian matrix

$$\det(e^H) = e^{tr(H)}$$

Example 5.

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (\sigma_3)^n = \begin{pmatrix} 1 & 0 \\ 0 & (-1)^n \end{pmatrix}$$

$$e^{\sigma_3} = \begin{pmatrix} \sum_{n=0}^{\infty} \frac{1}{n!} & 0 \\ 0 & \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \end{pmatrix} = \begin{pmatrix} e & 0 \\ 0 & e^{-1} \end{pmatrix}$$

Baker - Hausdorff formula.

$$e^{-T} A e^{T} = A + [A,T] + \frac{1}{2!}[[A,T],T] + \frac{1}{3!} + [[[A,T],T,T] + \cdots$$

Proof: $f(\lambda) = e^{\lambda A} B e^{-\lambda A}$

$f(0) = B$

$$f'(0) = \left(e^{\lambda A} A B e^{-\lambda A} + e^{\lambda A} B(-A) e^{-\lambda A}\right)_{\lambda=0} = e^{\lambda A}[A,B]e^{-\lambda A}\Big|_{\lambda=0} = [A,B]$$

$$f''(0) = \left[e^{\lambda A} A[A,B] e^{-\lambda A} + e^{\lambda A}[A,B](-A) e^{-\lambda A}\right]_{\lambda=0} = [A,[A,B]]$$

...

$$f^{(k)}(0) = [A,\cdot\,]^k B$$

$$f(\lambda) = \sum_{k=0}^{1} \frac{1}{k!} \lambda^k f^{(k)}(0) = \sum_{k=0}^{\infty} \frac{1}{k!} \lambda^k [A,\cdot\,]^k B = e^{\lambda[A,\cdot\,]} B$$

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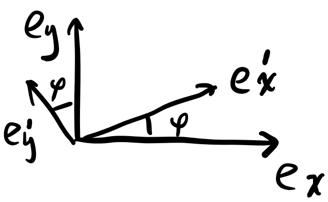
4. Coordinate Transformation



$$e_x = \cos\varphi\, e_x' - \sin\varphi\, e_y'$$
$$e_y = \sin\varphi\, e_x' + \cos\varphi\, e_y'$$

$$\vec{A} = A_x e_x + A_y e_y = (A_x \cos\varphi + A_y \sin\varphi) e_x' + (-A_x \sin\varphi + A_y \cos\varphi) e_y'$$

$$\begin{pmatrix} A_x' \\ A_y' \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix}$$

$$\begin{pmatrix} A_x \\ A_y \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} A_x' \\ A_y' \end{pmatrix}$$

ie $S = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$, $S' = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} = S^T$

$\vec{A} = S'S\vec{A} \Rightarrow S'S = \mathbb{1}$, S is orthogonal

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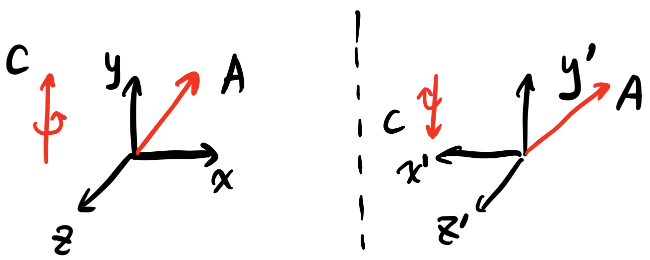
$$e_x = (e_x' \cdot e_x)e_x' + (e_y' \cdot e_x)e_y'$$
$$e_y = (e_x' \cdot e_y)e_x' + (e_y' \cdot e_y)e_y'$$

$$\Rightarrow S = \begin{pmatrix} e_x' \cdot e_x & e_x' \cdot e_y \\ e_x \cdot e_y' & e_y' \cdot e_y \end{pmatrix} = \begin{pmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial x}{\partial y'} & \frac{\partial y'}{\partial y} \end{pmatrix} = \frac{\partial x_\mu}{\partial x_\nu'}$$

$$e_x' \cdot e_x = \frac{\partial x'}{\partial x} = \frac{\partial x}{\partial x'}$$

4.1 reflection



$A' = SA$

$$\begin{cases} A_x' = -A_x \\ A_y' = A_y \\ A_z' = A_z \end{cases} \qquad \begin{cases} B_x' = -B_x \\ B_y' = B_y \\ B_z' = B_z \end{cases}$$

$C = A \times B$, $\quad C_x = A_yB_z - A_zB_y = C_x'$

$\downarrow$

pseudovector

$C_y = A_zB_x - A_xB_z = -C_y'$

$C_z = A_xB_y - A_yB_x = -C_z'$

$C' = \det(S)\, S\, C$

$$\begin{cases} \text{vector} & (V) \\ \text{pseudovector} & (P) \end{cases}$$

$P+P = P$, $V+V = V$, $P+V$

$$\vec{V_3} = \vec{V_1}+\vec{V_2} = S\vec{V_1}' + \det(S)\cdot S\vec{V_2}'$$

$$|\vec{V_3}| = |\vec{V_1}+\vec{V_2}| , \quad |\vec{V_3}'| = |\vec{V_1}'-\vec{V_2}'|$$

$V\times V = P$

$$\vec{V_3}' = (S\vec{V_1})\times(S\vec{V_2}) = (\det S)\, S\,(\vec{V_1}\times\vec{V_2}) = (\det S)\, S\,\vec{V_3}$$

$P\times P = P$

$$\vec{V_3}' = (\det S)(S\vec{V_1})\times(\det S)(S\vec{V_2}) = (\det S)^3\, S\,\vec{V_3}$$

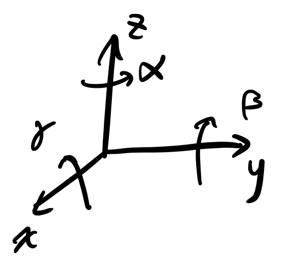
$P\times V = V$

$A\cdot B\times C$ pseudoscalar

$A\times(B\times C)$ is vector.

4.2. Rotations in $\mathbb{R}^3$.

$$S = \begin{pmatrix} e_1'\cdot e_1 & e_1'\cdot e_2 & e_1'\cdot e_3 \\ e_2'\cdot e_1 & e_2'\cdot e_2 & e_2'\cdot e_3 \\ e_3'\cdot e_1 & e_3'\cdot e_2 & e_3'\cdot e_3 \end{pmatrix}$$



$$S_1(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad S_2(\beta) = \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix}$$

$$S_3(\gamma) = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow S = S_3 S_2 S_1$$

$$S(\alpha,\beta,\gamma) = \begin{pmatrix} \cos\gamma\cos\beta\cos\alpha - \sin\gamma\sin\alpha & \cos\gamma\cos\beta\sin\alpha + \sin\gamma\cos\alpha & -\cos\gamma\sin\beta \\ -\sin\gamma\cos\beta\cos\alpha - \cos\gamma\sin\alpha & -\sin\gamma\cos\beta\sin\alpha + \cos\gamma\cos\alpha & \sin\gamma\sin\beta \\ \sin\beta\cos\alpha & \sin\beta\sin\alpha & \cos\beta \end{pmatrix}. \tag{3.37}$$

5. Differential Vector Operators.

5.1 Gradient

$$d\varphi = \frac{\partial\varphi}{\partial x_1}dx_1 + \frac{\partial\varphi}{\partial x_2}dx_2 + \frac{\partial\varphi}{\partial x_3}dx_3$$

$$\nabla\varphi = \left(\frac{\partial\varphi}{\partial x_1}, \frac{\partial\varphi}{\partial x_2}, \frac{\partial\varphi}{\partial x_3}\right) \quad \Rightarrow \quad \nabla\varphi\cdot d\vec{r} = d\varphi$$

$$S(\nabla\varphi) = \frac{\partial x_\mu}{\partial x_\nu'}\frac{\partial\varphi}{\partial x_\mu} = (\nabla\varphi)'$$

Example 6.

$$\nabla r^n = \frac{\partial r^n}{\partial r}\frac{\partial r}{\partial x_i}\vec{e_i} = n r^{n-1}\frac{x_i}{r}\vec{e_i} = n r^{n-1}\hat{r}$$

$$A\times(B\times C) = \varepsilon_{ijk}A_i\,\varepsilon_{lmj}B_lC_m\,\vec{e_k}$$

$$= \varepsilon_{kij}\varepsilon_{lmj}A_iB_lC_m\vec{e_k}$$

$$= (\delta_{kl}\delta_{im} - \delta_{km}\delta_{il})A_iB_lC_m\vec{e_k}$$

$$= (A_mB_kC_m - A_lB_lC_k)\vec{e_k}$$

$$\Rightarrow (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$$

5.2. Divergence

$$\nabla\cdot A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \qquad \nabla\cdot\vec{r} = \frac{\partial x_i}{\partial x_i} = 3$$

Example 7. $f(r)\hat{r}$

$$\nabla \cdot f(r)\,\hat{r} = \underline{\frac{\partial}{\partial x}\left(\frac{x f(r)}{r}\right)} + \frac{\partial}{\partial y}\left(\frac{y f(r)}{r}\right) + \frac{\partial}{\partial z}\left(\frac{z f(r)}{z}\right)$$

$$= \frac{f(r)}{r} - \frac{x f(r)}{r^2}\frac{\partial r}{\partial x} + \frac{x}{r}\frac{df(r)}{dr}\frac{\partial r}{\partial x} = f(r)\left[\frac{1}{r} - \frac{x^2}{r^3}\right] + \frac{x^2}{r^2}\frac{df(r)}{dr}$$

$$\Rightarrow \nabla \cdot f(r)\,\hat{r} = 2\frac{f(r)}{r} + \frac{df(r)}{dr}$$

Alternatively,

$$\nabla \cdot f(r)\hat{r} = \frac{1}{r^2 \cancel{\sin\theta}}\left(\frac{\partial}{\partial r}\left(r^2 \cancel{\sin\theta} f(r)\right)\right) = \frac{1}{r^2}\left(2r f(r) + \frac{\partial f(r)}{\partial r} r^2\right)$$

$$\nabla \cdot r^n \hat{r} = (n+2)\, r^{n-1}$$

5.3. equation of continuity

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \\ \dfrac{\partial w}{\partial t} + \nabla \cdot \vec{S} = 0 \\ \dfrac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0 \end{cases}$$

$\nabla \cdot B = 0 \Rightarrow$ Solenoidal.

5.4. curl

$$\nabla \times V = \begin{vmatrix} e_x & e_y & e_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & V_z \end{vmatrix}$$

Example 8. $\nabla \times [f(r)\,\hat{r}]_x$

$$= \frac{\partial}{\partial y}\frac{z f(r)}{r} - \frac{\partial}{\partial z}\frac{y f(r)}{r} = 0.$$

$$= \varepsilon_{ijk}\,\partial_i\, f(r)\frac{r_j}{r}\vec{e_k} \qquad \left(\partial_i = \frac{\partial}{\partial x_i}\right)$$

$$= \varepsilon_{ijk} r_j \frac{d}{dr}\left(\frac{f(r)}{r}\right) \frac{r_i}{r} \vec{e}_k = 0$$

$\nabla \times \vec{B} = 0 \Rightarrow$ irrotational.

5.5. Laplacian.

$$\nabla^2 \varphi = \nabla \cdot \nabla\varphi$$

$$\nabla^2 \vec{A} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial z^2}\right)(A_x, A_y, A_z) = (\nabla\cdot\nabla)\vec{A}$$

Example 9.

$$\nabla^2 \varphi(r) = \nabla \cdot \frac{d\varphi(r)}{dr} \vec{e}_r = 2\frac{d\varphi/dr}{r} + \frac{d^2\varphi}{dr^2}$$

$$\nabla^2 r^n = n(n+1) r^{n-2}$$

5.6. Two identrties

(i) $\nabla \times \nabla\varphi = 0$

$$= \varepsilon_{ijk} \partial_i \partial_j \varphi \vec{e}_k = 0$$

(ii) $\nabla \cdot (\nabla \times \vec{V}) = 0$

$$= \partial_i \varepsilon_{kji} \partial_k V_j$$

$$= \varepsilon_{kji} \partial_i \partial_k V_j = 0$$

Example 10.

$$\begin{cases} \nabla\cdot E = \rho/\varepsilon_0 \\ \nabla\times E = -\frac{\partial B}{\partial t} \\ \nabla\cdot B = 0 \\ \nabla\times B = \mu_0 J + \mu_0\varepsilon_0 \frac{\partial E}{\partial t} \end{cases}$$

$$\nabla\times(\nabla\times E) = -\frac{\partial}{\partial t}(\nabla\times B)$$

$$= -\mu_0 \frac{\partial J}{\partial t} - \mu_0\varepsilon_0 \frac{\partial^2 E}{\partial t^2}$$

$$LHS = \nabla(\nabla\cdot E) - \nabla^2 E$$

$$\Rightarrow \nabla^2 E - \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial J}{\partial t} + \nabla \frac{\rho}{\varepsilon_0}$$

5.7. Indentities.

(1) $\nabla(\vec{A}\cdot\vec{B}) = (\vec{B}\cdot\nabla)\vec{A} + (\vec{A}\cdot\nabla)\vec{B} + \vec{A}\times(\nabla\times\vec{B}) + \vec{B}\times(\nabla\times\vec{A})$

(2) $\nabla\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\nabla)\vec{A} - (\vec{A}\cdot\nabla)\vec{B} + \vec{A}(\nabla\cdot\vec{B}) - \vec{B}(\vec{\nabla}\cdot\vec{A})$

(1) $\nabla(\vec{A}\cdot\vec{B})$

$$= \nabla_A(\vec{A}\cdot\vec{B}) + \nabla_B(\vec{A}\cdot\vec{B})$$

$$= \vec{B}\times(\nabla\times\vec{A}) + (B\cdot\nabla)\vec{A} + A\times(\nabla\times B) + (\vec{A}\cdot\nabla)B$$

(2) $\nabla\times(\vec{A}\times\vec{B})$

$$= \nabla_A\times(\vec{A}\times\vec{B}) + \nabla_B(\vec{A}\times\vec{B})$$

$$= (\vec{B}\cdot\nabla)\vec{A} - (\nabla\cdot A)\vec{B} + (A\cdot\nabla)\vec{B} - (\nabla\cdot B)\vec{A}$$

5.8. Integral

Example 11. $\quad r\to\infty,\ f, A \longrightarrow 0$

$$\int \vec{A}(\vec{r})\cdot\nabla f(\vec{r})\, d^3\vec{r}$$

$$= \iint dy\,dz\left[A_x \frac{\partial f}{\partial x}\,dx\right] + \cdots$$

$$= \iint dy\,dz\left[A_x f\Big|_{-\infty}^{+\infty} - \int f\frac{\partial A_x}{\partial x}\,dx\right] + \cdots$$

$$= -\iiint dx\,dy\,dz \cdot f\frac{\partial A_x}{\partial x} + \cdots$$

$$= -\int f(r)\,\nabla\cdot\vec{A}\, d^3\vec{r}$$

$$\Rightarrow \int f(\vec r)\nabla\cdot\vec A \, d^3\vec r = -\int \vec A(\vec r)\cdot \nabla f(\vec r)\, d^3\vec r$$

$$\int \vec C(\vec r)\cdot\left(\nabla\times\vec A(\vec r)\right) d^3\vec r = \int \vec A(\vec r)\cdot\left(\nabla\times\vec C\right) d^3\vec r$$

6. Integral Theorem

6.1. Gauss' Theorem

$$\oint_{\partial V}\vec A\cdot d\vec\sigma = \int_V \nabla\cdot\vec A \, d\tau$$

Corollary: Green's Theorem

identies: $\nabla\cdot(u\nabla v) = u\nabla^2 v + (\nabla u)\cdot(\nabla v)$

$\nabla\cdot(v\nabla u) = v\nabla^2 u + (\nabla u)\cdot(\nabla v)$

1. $\int_V (u\nabla^2 v - v\nabla^2 u)\, d\tau = \oint_{\partial V}(u\nabla v - v\nabla u)\, d\sigma$

2. $\oint_{\partial V}(u\nabla v)\cdot d\vec\sigma = \int_V u^2\nabla^2 v \, d\tau + \int_V (\nabla\cdot u)(\nabla\cdot v)\, d\tau$

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a. $\vec B(x,y,z) = B(x,y,z)\,\vec a$

$$\oint_{\partial V} B(x,y,z)\,\vec a\cdot d\sigma = \int_V \nabla\cdot(B\vec a)\, d\tau$$

$$= \vec a\cdot\oint_{\partial V} B\, d\vec\sigma = \vec a\cdot\int_V \nabla B \, d\tau$$

$$\Rightarrow \oint_{\partial V} B\, d\sigma = \int_V \nabla B \, d\tau$$

b. $\vec B = \vec a\times\vec P$

$$\oint_{\partial V}(\vec a\times\vec P)\cdot d\vec\sigma = \int_V \nabla\cdot(\vec a\times\vec P)\, d\tau$$

$$= \left[\oint_{\partial V} \vec{P} \times d\vec{\sigma}\right] \cdot \vec{a} = \int_V -(\nabla \times \vec{P})\, d\tau \cdot \vec{a}$$

$$\Rightarrow \oint_{\partial V} d\vec{\sigma} \times \vec{P} = \int_V \nabla \times \vec{P}\, d\tau$$

6.2 Stoke's Theorem

$$\oint_{\partial S} \vec{B} \cdot d\vec{r} = \int_S (\nabla \times \vec{B}) \cdot d\vec{\sigma}$$

a.

$$\vec{a} \cdot \oint_{\partial S} \varphi\, d\vec{r} = \int_S (\nabla \times (\varphi \vec{a})) \cdot d\vec{\sigma}$$

$$= \vec{a} \cdot \int_S d\vec{\sigma} \times \nabla \varphi$$

$$\Rightarrow \oint_{\partial S} \varphi\, d\vec{r} = \int_S d\vec{\sigma} \times \nabla \varphi$$

b. $$\oint_{\partial S} (\vec{a} \times \vec{P}) \cdot d\vec{r} = \int_S (\nabla \times (\vec{a} \times \vec{P})) \cdot d\vec{\sigma}$$

$$= \vec{a} \oint_{\partial S} d\vec{r} \times \vec{P} = \vec{a} \cdot \int_S (d\sigma \times \nabla) \times \vec{P} \cdot$$

$$\Rightarrow \int_S (d\vec{\sigma} \times \nabla) \times \vec{P} = \oint_{\partial S} d\vec{r} \times \vec{P}$$

Example 12.

$$\begin{cases} \nabla \cdot E = \rho/\varepsilon_0 & ① \\ \nabla \times E = -\frac{\partial B}{\partial t} & ② \\ \nabla \cdot B = 0 & ③ \\ \nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} & ④ \end{cases} \Rightarrow \begin{cases} \oint_{\partial V} \vec{E} \cdot d\vec{S} = \int_V \rho/\varepsilon_0 \\ \oint_{\partial S} \vec{E} \cdot d\vec{l} = \int_S -\frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} \\ \oint_{\partial V} \vec{B} \cdot d\vec{S} = 0 \\ \oint_{\partial S} \vec{B} \cdot d\vec{l} = \mu_0 I + \mu_0 \varepsilon_0 \int_S \frac{\partial E}{\partial t} d\vec{S} \end{cases}$$

7. Potential Theory.

③ $\Rightarrow \quad \vec{B} = \nabla \times \vec{A}$

② $\Rightarrow \quad \nabla \times E = -\frac{\partial}{\partial t} \nabla \times A \Rightarrow \nabla \times \left(E + \frac{\partial A}{\partial t} \right) = 0$

$\Rightarrow \quad E + \frac{\partial A}{\partial t} = \nabla \varphi \quad \Rightarrow \quad E = -\nabla \varphi - \frac{\partial \vec{A}}{\partial t}$

Lorentz Gauge : $\frac{1}{c^2} \frac{\partial \varphi}{\partial t} + \nabla \cdot \vec{A} = 0$

$$\frac{\rho}{\varepsilon_0} = \nabla \cdot E = -\nabla^2 \varphi - \frac{\partial}{\partial t} \nabla \cdot A = -\nabla^2 \varphi + \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2}$$

$$\nabla (\nabla \times \vec{A}) = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \left(-\nabla \varphi - \frac{\partial \vec{A}}{\partial t} \right)$$

$$= \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \left[-\nabla \frac{\partial}{\partial t} \varphi - \frac{\partial \vec{A}}{\partial t} \right]$$

$$\Rightarrow \begin{cases} \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} = \mu_0 \vec{J} \\ \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} - \nabla^2 \varphi = \frac{\rho}{\varepsilon_0} \end{cases}$$

$$\begin{cases} \nabla^2 \varphi = -\frac{\rho}{\varepsilon_0} \\ \nabla^2 \varphi = 0 \\ \nabla^2 \varphi = -\frac{q}{\varepsilon_0} \delta(\vec{r}) \end{cases}$$

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8. Helmholtz Theorem

$$\nabla \cdot \vec{F}(\vec{r}) = D(\vec{r})$$
$$\nabla \times \vec{F}(\vec{r}) = \vec{C}(\vec{r})$$

则在 $\vec{F}(\vec{r}, t) \xrightarrow{|r| \to \infty} \frac{1}{|r|^{1+\delta}} \to 0, \ \delta > 0$.

有 $\vec{F}(\vec{r}) = -\frac{1}{4\pi} \nabla \int_{全} d^3\vec{r}' \frac{D(\vec{r}')}{|\vec{r} - \vec{r}'|} + \frac{1}{4\pi} \nabla \times \int_{全} d^3\vec{r}' \frac{\vec{C}(\vec{r}')}{|\vec{r} - \vec{r}'|}$.

Proof: $F(\vec{r})$

$$\vec{W}(\vec{r}) = \frac{1}{4\pi}\int_{全} d^3\vec{r}' \frac{\vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$\nabla^2 \vec{W}(\vec{r}) = \frac{1}{4\pi}\int_{全} d^3r' \vec{F}(\vec{r}') \nabla^2\frac{1}{|\vec{r}-\vec{r}'|} = -\int_{全} d^3\vec{r}' \vec{F}(\vec{r}')\delta(\vec{r}-\vec{r}')$$

$$= -\vec{F}(\vec{r})$$

$$\Rightarrow \vec{F}(\vec{r}) = -\nabla^2\vec{W}(\vec{r}) = -\left[\nabla(\nabla\cdot\vec{W}) - \nabla\times(\nabla\times\vec{W})\right]$$

$$-\nabla\cdot\vec{W}(\vec{r}) = -\frac{1}{4\pi}\int_{全} d^3\vec{r}' \vec{F}(\vec{r}')(-1)\nabla'\frac{1}{|\vec{r}-\vec{r}'|}$$

$$= \frac{1}{4\pi}\int_{全} d^3r'\left[\nabla'\cdot\frac{\vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|} - \frac{1}{|\vec{r}-\vec{r}'|}\nabla'\cdot\vec{F}(\vec{r}')\right]$$

$$= \frac{1}{4\pi}\int_{\partial全} d\vec{a}'\cdot\frac{\vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|} - \frac{1}{4\pi}\int_{全} d^3\vec{r}' \frac{\nabla'\cdot\vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$= -\frac{1}{4\pi}\int_{全} d^3\vec{r}' \frac{\nabla'\cdot\vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$\nabla\times\vec{W}(\vec{r}) = -\frac{1}{4\pi}\int_{全} d^3r' \nabla\times\left(\vec{F}(\vec{r}')\frac{1}{|\vec{r}-\vec{r}'|}\right)$$

$$= -\frac{1}{4\pi}\int_{全} d^3\vec{r}' \nabla'\frac{1}{|\vec{r}-\vec{r}'|}\times\vec{F}(\vec{r}')$$

$$= -\frac{1}{4\pi}\int d^3r'\left(\nabla'\times\left(\frac{\vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right) - \frac{\nabla'\times\vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right)$$

$$= -\frac{1}{4\pi}\int_{全} d\vec{a}\times\frac{\vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|} + \frac{1}{4\pi}\int_{全} d^3r' \frac{\nabla'\times\vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$= \frac{1}{4\pi}\int_{全} d^3r' \frac{\nabla'\times\vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|}, \quad \text{Q.E.D.}$$

9. Curvilinear Coordinate (orthogonal)

$$\vec{V}(\vec{r}) = V_r\hat{e}_r + V_\theta\hat{e}_\theta + V_\varphi\hat{e}_\varphi$$

$$dx = \frac{\partial x}{\partial q_1} dq_1 + \frac{\partial x}{\partial q_2} dq_2 + \frac{\partial x}{\partial q_3} dq_3$$

$$dr^2 = dx^2 + dy^2 + dz^2$$

$$= \left(\frac{\partial x}{\partial q_i} dq_i\right)\left(\frac{\partial x}{\partial q_j} dq_j\right) + \cdots$$

$$= \left(\frac{\partial x}{\partial q_i} \frac{\partial x}{\partial q_j} dq_i dq_j\right) + \cdots$$

$$= g_{ij} \, dq_i \, dq_j$$

$$g_{ij} = \frac{\partial x_k}{\partial q_i} \frac{\partial x_k}{\partial q_j}$$ Metric Tensor (度规)

Specilize it,

$$dr^2 = (h_1 dq_1)^2 + (h_2 dq_2)^2 + (h_3 dq_3)^2$$

$$h_i = \left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial q_i}\right)^2$$

$$\Rightarrow dr_i = h_i dq_i \Rightarrow h_i \hat{e}_i = \frac{\partial \vec{r}}{\partial q_i}$$

$$\Rightarrow d\vec{r} = h_1 dq_1 \hat{e}_1 + h_2 dq_2 \hat{e}_2 + h_3 dq_3 \hat{e}_3$$

h 拉梅系数

$$\nabla = \hat{e}_1 \frac{1}{h_1} \frac{\partial}{\partial q_1} + \hat{e}_2 \frac{1}{h_2} \frac{\partial}{\partial q_2} + \hat{e}_3 \frac{1}{h_3} \frac{\partial}{\partial q_3}$$

$$\nabla \cdot V = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1}(V_1 h_2 h_3) + \frac{\partial}{\partial q_2}(V_2 h_1 h_3) + \frac{\partial}{\partial q_3}(V_3 h_1 h_2) \right.$$

$$\nabla \times B = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \hat{e}_1 h_1 & \hat{e}_2 h_2 & \hat{e}_3 h \\ \partial/\partial q_1 & \partial/\partial q_2 & \partial/\partial q_3 \\ h_1 B_1 & h_2 B_2 & h_3 B_3 \end{vmatrix}$$

$$\nabla^2 \varphi(q_1, q_2, q_3) = \nabla \cdot \nabla \varphi$$

$$= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\varphi}{\partial q_1}\right)+\frac{\partial}{\partial q_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\varphi}{\partial q_2}\right)+\frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\varphi}{\partial q_3}\right)\right]$$

习题：

1. $A \cdot B$ two noncommuting Hermitian matrices.

$$AB - BA = iC$$

Prove C is Hermitian (2.2.3b)

2. (i) $\vec{r}' = U\vec{r}$, U is unitary, r is a vector with complex number, show $|r|$ is invariant.

(ii) $U: \vec{r} \to \vec{r}'$, $r^\dagger r = r'^\dagger r$ $\Rightarrow$ U is unitary

(2.2.51)

3. $\hat{L} = -i(\vec{r}\times\nabla)$

(a) $L_x + iL_y = e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)$

(b) $L_x - iL_y = -e^{i\varphi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\varphi}\right)$

(c) $[L_i, L_j] = i\varepsilon_{ijk}L_k$ (3.10.31)