张量分析与微分形式

1. Tensor Analysis

1.1 Introductor

scalar — 0 rank

vector — 1 rank

A tensor of rank n, d-dimension

1. n indices, $n = 1 \sim d$, d^n components
2. component transformed in a special manner

1.2 Covariant and Contravariant tensor

$$\begin{cases} A_i' = \sum_i (\hat{e}_i' \cdot \hat{e}_j) A_j = \sum_j \frac{\partial x_i'}{\partial x_j} A_j \\ (\nabla\varphi)_i' = \frac{\partial \varphi}{\partial x_i'} = \sum_j \frac{\partial x_j}{\partial x_i'} \frac{\partial \varphi}{\partial x_j} \end{cases}$$

$$A'^i = \frac{\partial x'^i}{\partial x^j} A^j \qquad \text{contravariant}$$

$$B_i' = \frac{\partial x^j}{\partial x'^i} B_j \qquad \text{covariant}$$

1.3 Tensor of rank 2

$$(A')^{ij} = \frac{\partial x'^i}{\partial x^k} \cdot \frac{\partial x'^j}{\partial x^l} A^{kl}$$

$$(A')^i{}_j = \frac{\partial x'^i}{\partial x^k} \cdot \frac{\partial x^l}{\partial x'^j} A^k{}_l$$

$$(A')_{ij} = \frac{\partial x^k}{\partial x'^i} \cdot \frac{\partial x^l}{\partial x'^j} A_{kl}$$

$$(A')^{ij} = S_{ik} A^{kl} S^T_{lj}$$

$$\Rightarrow A' = S A S^T$$

Similarity transformation

Congruent — —

1.4 $A + B = C$

$$A^{ij} + B^{ij} = C^{ij}$$

Symmetric $A^{mn} = A^{nm}$ $\quad A^{mn} = \frac{1}{2}(A^{mn} + A^{nm}) + \frac{1}{2}(A^{mn} - A^{nm})$

anti-symmetry $A^{mn} = -A^{nm}$

1.5 Isotropic tensor

$$\delta^k{}_l \frac{\partial x'^i}{\partial x^k} \cdot \frac{\partial x^l}{\partial x'^j} = \frac{\partial x'^i}{\partial x^l} \cdot \frac{\partial x^l}{\partial x'^j} = \frac{\partial x'^i}{\partial x'^j} = \delta'^i{}_j$$

1.6. Contraction.

$$\vec{A} \cdot \vec{B} = A_i B^i$$

$$(B')^i{}_i = \frac{\partial x'^i}{\partial x^k} \cdot \frac{\partial x^l}{\partial x'^i} B^k{}_l = \frac{\partial x^l}{\partial x^k} B^k_l = \delta^l_k B^k{}_l = B^k{}_k$$

scalar, trace is invariant.

$$tr(P^{-1}AP) = tr(APP^{-1}) = tr(A)$$

1.7. Direct Product.

$$A^i{}_k B^j{}_{lm} = C^{ij}{}_{klm}, \quad A^j B^i{}_{kl} = F^{ij}_{kl}$$

Example 1.

$$C'_i{}^j = a'_i b'^j = \frac{\partial x^k}{\partial x'_i} a_k \frac{\partial x'^j}{\partial x^l} b^l = \frac{\partial x^k}{\partial x'^i} \frac{\partial x'^j}{\partial x^l} C^l_k$$

Generally. $\frac{\partial x^i}{\partial x'^j} \neq \left(\frac{\partial x'^j}{\partial x^i}\right)^{-1}$.

1.8 Quotient rule

$$K_i A^i = B \quad , \quad K_{iljk} A^{ik} = B_{kl}$$

if the equation holds in all transformed coordinate system.

$$K_i{}^j A_j = B_i \longrightarrow K'_i{}^j A'_j = B_i'$$

$$B_i' = \frac{\partial x^m}{\partial x'^i} B_m = \frac{\partial x^m}{\partial x'^i} K_m{}^j A_j = \frac{\partial x^m}{\partial x'^i} K_m{}^j \frac{\partial x'^n}{\partial x^j} A'_n$$

$$= \frac{\partial x^m}{\partial x'^i} \cdot \frac{\partial x'^n}{\partial x^j} K_m{}^j A'_n = (K')_i{}^j A'_j$$

$$K'_i{}^j = \frac{\partial x^m}{\partial x'^i} \cdot \frac{\partial x'^n}{\partial x^j} K_m{}^j$$

Example 2.

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right] A^\mu = J^\mu$$

1.9. Spinor

Spin	0	1	2	1/2
	↓	↓	↓	↓
	scalar	vector	tensor	Spinor

1.10 pseudo vector

$$A' = SA \quad , \quad A' = \det(S)\, S A$$

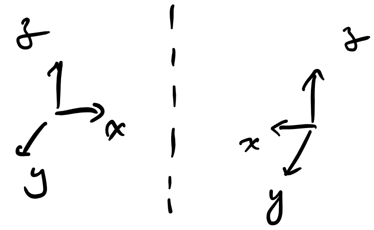
vector   ↑ pseudo vector

$$V \times V = P \times P = P$$

$$T \times P = P \times T = T$$

Example 3. 三阶全反对称不变赝张量

$$\eta_{ijk} = \det(A)\, a_{ip}\, a_{jq}\, a_{kr}\, \varepsilon_{pqr}$$



1.11 Dual Tensor

Anti-symmetric tensor C, associate with a pseudovector.

second rank

$$C_i = \frac{1}{2}\varepsilon_{ijk} C^{jk} \qquad (C_1, C_2, C_3) = (C^{23}, C^{31}, C^{12})$$

$$C = \begin{pmatrix} 0 & C^{12} & -C^{31} \\ -C^{12} & 0 & C^{23} \\ C^{31} & -C^{23} & 0 \end{pmatrix}$$

different representation of the same thing.

$$V^{ijk} = A^i B^j C^k$$

$$V = \varepsilon_{ijk} V^{ijk} = \varepsilon_{ijk} A^i B^j C^k = \begin{vmatrix} A^1 & B^1 & C^1 \\ A^2 & B^2 & C^2 \\ A^3 & B^3 & C^3 \end{vmatrix}$$

Hodge dual. n-dimension space. p-rank tensor

$$\varepsilon : T_{i_1 \cdots i_p} \longmapsto \frac{1}{(n-p)!}\, \varepsilon_{i_1 \cdots i_n} T_{i_{n-p+1} \cdots i_n}$$

2. Tensor in general coordinate.

2.1 Metric tensor

2.1.1 covariant basis vector $\hat{\varepsilon}_i$

$$\hat{\varepsilon}_i = \frac{\partial x}{\partial q^i}\hat{e}_x + \frac{\partial y}{\partial q^i}\hat{e}_y + \frac{\partial z}{\partial q^i}\hat{e}_z$$

$$\vec{A} = A^1\hat{\varepsilon}_1 + A^2\hat{\varepsilon}_2 + A^3\hat{\varepsilon}_3$$

$$ds^2 = (\hat{\varepsilon}_i\, dq_i)\cdot(\hat{\varepsilon}_j \cdot dq_j) = \hat{\varepsilon}_i \cdot \hat{\varepsilon}_j\, dq_i\, dq_j = g_{ij}\, dq_i\, dq_j$$

Define $g_{ij} = \hat{\varepsilon}_i \cdot \hat{\varepsilon}_j$

Covariant tensor

$$g^{ik} g_{kj} = g_{jk} g^{ki} = \delta^i{}_j$$

$$g_{ij} F^j = F_i \quad , \quad g^{ij} F_j = F^i$$

$$\vec{A} = A^i \hat{\varepsilon}_i = A^i \delta_i{}^k \hat{\varepsilon}_k = A^i g_{ij} g^{jk} \hat{\varepsilon}_k = A_j \hat{\varepsilon}^j$$

2.1.2 Contravariant Bases

$$\hat{\varepsilon}^i = \frac{\partial q^i}{\partial x}\hat{e}_x + \frac{\partial q^i}{\partial y}\hat{e}_y + \frac{\partial q^i}{\partial z}\hat{e}_z$$

$$\hat{\varepsilon}^i \cdot \hat{\varepsilon}_j = \frac{\partial q^i}{\partial x}\frac{\partial x}{\partial q_j} + \cdots = \frac{\partial q^i}{\partial x^k}\frac{\partial x^k}{\partial q^j} = \delta_{ij}$$

$$(\hat{\varepsilon}^i \cdot \hat{\varepsilon}^j)(\hat{\varepsilon}_j \cdot \hat{\varepsilon}_k)$$

$$= \left(\frac{\partial q^i}{\partial x^k}\boxed{\frac{\partial q^j}{\partial x^k}\right)\left(\frac{\partial x^l}{\partial q^j}}\frac{\partial x^l}{\partial q^k}\right) = \frac{\partial q^i}{\partial x^l}\frac{\partial x^l}{\partial q^k} = \delta^i{}_k$$

$$\Rightarrow \quad \hat{\varepsilon}^i \cdot \hat{\varepsilon}^j = g^{ij} \quad \Rightarrow \quad g^{ij}\hat{\varepsilon}_j = \hat{\varepsilon}^i$$

Example 4.

$$x = r\sin\theta\cos\varphi \qquad y = r\sin\theta\sin\varphi \qquad z = r\cos\theta$$

$$\hat{\varepsilon}_r = (\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta)$$

$$\hat{\varepsilon}_\theta = (r\cos\theta\cos\varphi,\ r\cos\theta\sin\varphi,\ -r\sin\theta)$$

$$\hat{\varepsilon}_\varphi = (-r\sin\theta\sin\varphi,\ r\sin\theta\cos\varphi,\ 0)$$

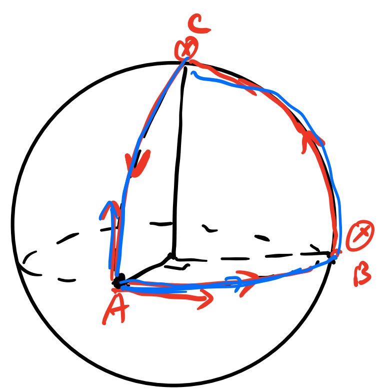
$$g_{ij} = \begin{pmatrix} 1 & & \\ & r^2 & \\ & & r^2\sin^2\theta \end{pmatrix} \qquad g^{ij} = \begin{pmatrix} 1 & & \\ & r^{-2} & \\ & & r^{-2}\sin^{-2}\theta \end{pmatrix}$$

$$\vec{A}\cdot\vec{B} = (A^i\hat{\varepsilon}_i)\cdot(B_j\hat{\varepsilon}^j) = A^iB_j(\hat{\varepsilon}_i\cdot\hat{\varepsilon}^j) = A^iB_i$$

$$(\nabla\varphi)_j = \frac{\partial\varphi}{\partial q^i}\frac{\partial q^i}{\partial x^j}\hat{e}_j = \frac{\partial\varphi}{\partial q^i}\hat{\varepsilon}^i = \frac{\partial\varphi}{\partial q^i}g^{il}\hat{\varepsilon}_l$$

$$= \frac{\partial\varphi}{\partial q_l}\hat{\varepsilon}_l$$

2.2. Covariant Derivative



$$(V')^i = \frac{\partial x^i}{\partial q^k}V^k$$

$$\frac{\partial V'^i}{\partial q^j} = \frac{\partial x^i}{\partial q^k}\frac{\partial V^k}{\partial q^j} + \frac{\partial^2 x^i}{\partial q^j\partial q^k}V^k$$

$$\frac{\partial \vec{V}'}{\partial q^j} = \frac{\partial V^k}{\partial q^j}\hat{\varepsilon}_k + V^k \frac{\partial \hat{\varepsilon}_k}{\partial q^j}$$

Let $\frac{\partial \hat{\varepsilon}_k}{\partial q^j} = \Gamma^{\mu}_{jk}\hat{\varepsilon}_\mu$, Γ^{μ}_{jk} Christoffel symbol of the second kind

$$\Gamma^m_{jk} = \hat{\varepsilon}^m \cdot \frac{\partial \hat{\varepsilon}_k}{\partial q^j}$$

$$\Gamma^m_{jk} = \Gamma^m_{kj}$$

Proof : $\frac{\partial \hat{\varepsilon}_k}{\partial q^j} = \frac{\partial}{\partial q^j}\left(\frac{\partial x^l}{\partial q^k}\hat{e}_l\right)$

$$\frac{\partial \hat{\varepsilon}_j}{\partial q^k} = \frac{\partial}{\partial q^k}\left(\frac{\partial x^l}{\partial q^j}\hat{e}_l\right)$$

$$\frac{\partial \vec{V}'}{\partial q^j} = \frac{\partial V^k}{\partial q^j}\hat{\varepsilon}_k + V^\mu \Gamma^k_{j\mu}\hat{\varepsilon}_k$$

$$= \left(\frac{\partial V^k}{\partial q^j} + V^\mu \Gamma^k_{j\mu}\right)\hat{\varepsilon}_k$$

$$V^k_{;j} = \frac{\partial V^k}{\partial q^j} + V^\mu \Gamma^k_{j\mu} = V^k_{,j} + V^\mu \Gamma^k_{j\mu}$$

$\Rightarrow \nabla_j \vec{V}' = V^k_{;j}\,\hat{\varepsilon}_k$

$\frac{\partial V^k}{\partial q^j}$, $\Gamma^k_{j\mu}$ isn't

$V^k_{;j}$ is.

$$d\vec{V}' = \left[V^k_{;j}\,dq^j\right]\hat{\varepsilon}_k$$

$$\partial_i(U_m U^m) = (\partial_i U_m) U^m + U_m \partial_i U^m$$

$$\Rightarrow (\partial_i U_m) U^m = \partial_i (U_m U^m) - U_m \partial_i U^m$$

$$(\nabla_i U_m) U^m = \partial_i (U_m U^m) - U_m \nabla_i U^m$$

$$= \partial_i (U_m U^m) - U_m \left(\partial_i U^m + \Gamma_{ij}^m U^j \right)$$

$$= (\partial_i U_m) U^m - \Gamma_{ij}^p U_p U^m$$

$$\Rightarrow \nabla_i U_m = \partial_i U_m - \Gamma_{im}^p U_p$$

2.3. Christoffel symbol of the first kind

$$[ij,k] = g_{mk} \Gamma_{ij}^m = \Gamma_{ij,k}$$

$$= g_{mk} \hat{\varepsilon}^m \cdot \frac{\partial \hat{\varepsilon}_i}{\partial q^j} = \hat{\varepsilon}_k \cdot \frac{\partial \hat{\varepsilon}_i}{\partial q^j}$$

$$g_{ij} = \hat{\varepsilon}_i \cdot \hat{\varepsilon}_j$$

$$\frac{\partial g_{ij}}{\partial q^k} = \frac{\partial \hat{\varepsilon}_i}{\partial q^k} \cdot \hat{\varepsilon}_j + \hat{\varepsilon}_i \cdot \frac{\partial \hat{\varepsilon}_j}{\partial q^k} = [ik,j] + [jk,i]$$

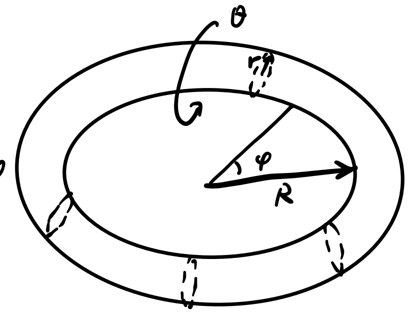
$$\frac{\partial g_{ik}}{\partial q^j} = [ij,k] + [jk,i]$$

$$\frac{\partial g_{jk}}{\partial q^i} = [ji,k] + [ki,j]$$

$$[ij,k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial q^j} + \frac{\partial g_{jk}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^k} \right)$$

$$\Rightarrow \Gamma^{n}_{ij} = g^{nk}[ij,k] = \frac{1}{2} g^{nk}\left(\frac{\partial g_{ik}}{\partial q^j} + \frac{\partial g_{jk}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^k}\right)$$

Example 5.



$$\frac{\partial x}{\partial \theta} = -r\sin\theta\cos\varphi \qquad \frac{\partial x}{\partial \varphi} = -(R+r\cos\theta)\sin\varphi$$

$$\frac{\partial y}{\partial \theta} = -r\sin\theta\sin\varphi \qquad \frac{\partial y}{\partial \varphi} = (R+r\cos\theta)\cos\varphi$$

$$\frac{\partial z}{\partial \theta} = r\cos\theta \qquad \frac{\partial z}{\partial \varphi} = 0$$

$$x = (R+r\cos\theta)\cos\varphi$$

$$y = (R+r\cos\theta)\sin\varphi$$

$$z = r\sin\theta$$

$$H_\theta = r^2, \quad H_\varphi = (R+r\cos\theta)^2$$

$$g_{\theta\phi} = g_{\phi\theta} = 0$$

$$g_{\mu\nu} = \begin{pmatrix} r^2 & 0 \\ 0 & (R+r\cos\theta)^2 \end{pmatrix}$$

$$\Gamma^{\alpha}_{\mu\nu} = \begin{pmatrix} (\cdot,\cdot) & (\cdot,\cdot) \\ (\cdot,\cdot) & (\cdot,\cdot) \end{pmatrix}$$

$$\Gamma^{n}_{ij} = \frac{1}{2} g^{nk}\left(\frac{\partial g_{ik}}{\partial q^j} + \frac{\partial g_{jk}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^k}\right)$$

$$\Gamma^{\theta}_{\theta\theta} = \frac{1}{2} g^{\theta k}\left(g_{\theta k,\theta} + g_{\theta k,\theta} - g_{\theta\theta,k}\right) = 0$$

$$\Gamma^{\theta}_{\theta\varphi} = 0 \qquad \Gamma^{\theta}_{\varphi\theta} = 0$$

$$\Gamma^{\theta}_{\varphi\varphi} = \frac{1}{2} g^{\theta k}\left(g_{\varphi k,\varphi} + g_{\varphi k,\varphi} - g_{\varphi\varphi,k} \right)$$

$$= \frac{1}{2} g^{\theta\theta} \left(- g_{\varphi\varphi,\theta} \right)$$

$$= \frac{1}{2} \frac{1}{(R + r\cos\theta)^2} \left(-2(R + r\cos\theta)\sin\theta\, r \right)$$

$$= \frac{r\sin\theta}{(R + r\cos\theta)}$$

$\Gamma^{\varphi}_{\theta\theta} =$ $\Gamma^{\varphi}_{\varphi\theta} =$

$\Gamma^{\varphi}_{\theta\varphi} =$ $\Gamma^{\varphi}_{\varphi\varphi} =$

$$\nabla_\nu A_\mu = \partial_\nu A_\mu - \Gamma^{\alpha}_{\mu\nu} A_\alpha = 0$$

$$\partial_\nu A_\mu = \Gamma^{\alpha}_{\mu\nu} A_\alpha$$

Example 6.

$\hat{\varepsilon}_r = (\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta)$ $\quad \frac{\partial \hat{\varepsilon}_r}{\partial r} = 0,\ \frac{\partial \hat{\varepsilon}_r}{\partial \theta} = \cdots,\ \frac{\partial \hat{\varepsilon}_r}{\partial \varphi} = \cdots$

$\hat{\varepsilon}_\theta = (r\cos\theta\cos\varphi,\ r\cos\theta\sin\varphi,\ -r\sin\theta)$ — — —

$\hat{\varepsilon}_\varphi = (-r\sin\theta\sin\varphi,\ r\sin\theta\cos\varphi,\ 0)$ — — —

$\hat{\varepsilon}^r = g^{r\beta}\varepsilon_\beta = g^{rr}\varepsilon_r = (\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta)$

$$\hat{\varepsilon}^\theta = g^{\theta\theta}\varepsilon_\theta = \left(\frac{1}{r}\cos\theta\cos\varphi, \frac{1}{r}\cos\theta\sin\varphi, -\frac{1}{r}\sin\theta\right)$$

$$\hat{\varepsilon}^\varphi = g^{\varphi\varphi}\varepsilon_\varphi = \frac{1}{r^2\sin^2\theta}\left(-r\sin\theta\sin\varphi, r\sin\theta\cos\varphi, 0\right)$$

2.4. Tensor Derivative Operator

Gradient $\nabla\varphi = \frac{\partial\varphi}{\partial q^i}\hat{\varepsilon}^i$

Divergence $$\nabla\cdot\vec{V} = \hat{\varepsilon}^j \cdot \frac{\partial(V^i\hat{\varepsilon}_i)}{\partial q^j} = \hat{\varepsilon}^j\left(\frac{\partial V^i}{\partial q^j} + \Gamma_{jk}^{\ \ i} V_k\right)\cdot\hat{\varepsilon}_i$$

$$= \frac{\partial V^i}{\partial q^i} + \Gamma_{ik}^{i} V_k$$

$$= \frac{\partial V^i}{\partial q^i} + V_k \cdot \frac{1}{2} g^{im}\left(g_{im,k} + \cancel{g_{km,i} - g_{ik,m}}\right)$$

$$= \frac{\partial V^i}{\partial q^i} + \frac{1}{2} V_k g^{im}\frac{\partial g_{im}}{\partial q^k}$$

$$\frac{\partial|g|}{\partial q^k} = |g|\, g^{im}\frac{\partial g_{im}}{\partial q^k}, \quad \Gamma_{ik}^{i} = \frac{1}{\sqrt{|g|}}\frac{\partial\sqrt{|g|}}{\partial q^k}$$

$$\Rightarrow \nabla\cdot\vec{V} = V^i_{;i} = \frac{\partial V^i}{\partial q^i} + \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial q^k}\left(\sqrt{|g|}\right)V^k$$

$$= \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial q^k}\left(\sqrt{|g|}\, V^k\right)$$

Laplacian
$$\nabla^2\varphi = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial q^k}\left(\sqrt{|g|}\, g^{ki}\frac{\partial\varphi}{\partial q^i}\right)$$

For orthogonal frame,

$$\nabla^2 \varphi = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial q^i}\left(\frac{h_1 h_2 h_3}{h_i^2}\frac{\partial \varphi}{\partial q^i}\right)$$

Curl

$$\frac{\partial V_i}{\partial q^j} - \frac{\partial V_j}{\partial q^i} = V_{i,j} - V_k \Gamma^k_{ij} - V_{j,i} + V_k \Gamma^k_{ji}$$

$$= V_{i;j} - V_{j;i}$$

Jacobian

$$\begin{pmatrix} \frac{ds}{du_1} \\ \vdots \\ \frac{ds}{du_n} \end{pmatrix} = \begin{pmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_2}{\partial u_1} & \frac{\partial x_3}{\partial u_1} & \cdots \\ & \ddots & & \\ & & \ddots & \\ & & & \ddots \end{pmatrix} \begin{pmatrix} \hat{e}_1 \\ \vdots \\ \hat{e}_n \end{pmatrix}$$

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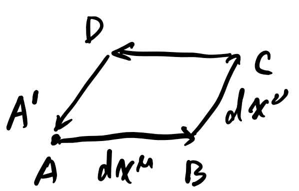
3. Introduction to GR



$$dV = V_A - V_{A'}$$

$$= [(V_C - V_D) - (V_B - V_A)] - [(V_C - V_B) - (V_D - V_{A'})]$$

$$= \nabla_\nu dx^\nu \nabla_\mu dx^\mu V - \nabla_\mu dx^\mu \nabla_\nu dx^\nu V$$

$$= dx^\mu dx^\nu [\nabla_\nu, \nabla_\mu] V$$

$[\nabla_\nu, \nabla_\mu]$

$$= (\partial_\nu + \Gamma_\nu)(\partial_\mu + \Gamma_\mu) - (\partial_\mu + \Gamma_\mu)(\partial_\nu + \Gamma_\nu)$$

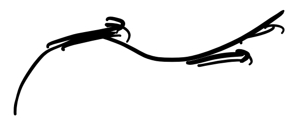
$$= (\partial_\nu \partial_\mu + \underline{\Gamma_\nu \partial_\mu} + \underline{\partial_\nu \Gamma_\mu} + \Gamma_\nu \Gamma_\mu) -$$

$$(\partial_\mu \partial_\nu + \partial_\mu \Gamma_\nu + \underline{\Gamma_\mu \partial_\nu} + \Gamma_\mu \Gamma_\nu)$$

$$= -[\partial_\mu, \Gamma_\nu] + [\partial_\nu \Gamma_\mu] + [\Gamma_\nu, \Gamma_\mu]$$

$$= -\frac{\partial \Gamma_\nu}{\partial x^\mu} + \frac{\partial F_\mu}{\partial x^\nu} + [\Gamma_\nu, \Gamma_\mu] = R_{\mu\nu} \leftarrow \text{Ricci 张量}$$

$$\nabla_\tau \frac{\partial x^\mu}{\partial \tau} = 0$$



$$\Rightarrow \frac{\partial}{\partial \tau} \frac{\partial x^\mu}{\partial \tau} + \Gamma = 0$$

$$\frac{\partial^2 x^\mu}{\partial \tau^2} = -\Gamma \sim F$$

$$\Gamma = \frac{1}{2} g^{\theta k} (g_{\varphi k,\varphi} + g_{\varphi k,\varphi} - g_{\varphi\varphi,k})$$

$$\frac{1}{2} \frac{\partial g^{00}}{\partial x} \sim F = -\frac{\partial \phi}{\partial x}$$

$$\Rightarrow g^{00} = 2\phi$$

$$\nabla^2 \phi = 4\pi G \rho$$

$$\Rightarrow \nabla^2 (g_{00}) = 8\pi G \rho$$

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$R_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\nabla_j R_{\mu\nu} \neq 0 \quad , \quad \nabla_j T_{\mu\nu} = 0$$

$$\nabla_\nu \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = 0$$

Proof: $[\nabla_\mu, [\nabla_\nu, \nabla_\kappa]] + [\nabla_\nu, [\nabla_\kappa, \nabla_\mu]] + [\nabla_\kappa, [\nabla_\mu, \nabla_\nu]] = 0$

Bianchi identity

$$R_{abmn;l} + R_{ablm;n} + R_{abnl;m} = 0$$

$$g^{bn} g^{am} \left(R_{abmn;l} + R_{ablm;n} + R_{abnl;m} \right) = 0$$

$$g^{bn} \left(R^m{}_{bmn;l} - R^m{}_{bml;n} + R^m{}_{bnl;m} \right) = 0$$

$$\Rightarrow \quad R^n{}_{n;l} - R^n{}_{l;n} - R^{nm}{}_{nl;m} = 0$$

$$\Rightarrow \quad R_{;l} - R^n{}_{l;n} - R^m{}_{l;m} = 0$$

$$\Rightarrow \quad R_{;l} = 2R^m{}_{l;m}$$

$$\Rightarrow \quad \nabla_l R^l{}_m = \frac{1}{2} \nabla_l g_{ml} R$$

$$\Rightarrow \quad \nabla_\nu \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} \right) = 0$$

$$\Rightarrow \quad \boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G T_{\mu\nu}}{c^4}}$$

2. Differential Forms

2.1. Introduction $\quad dx, dy, dz$

$w = A\,dx + B\,dy + C\,dz \quad$ 1- form

$w = F\,dx\wedge dy + G\,dx\wedge dz + H\,dy\wedge dz \quad$ 2- form

$w = K\,dx\wedge dy\wedge dz \quad$ 3- form

exterior algebra (Grassmann Algebra)

p - forms, p factors dx_i,

3. Exterior algebra $\quad \wedge$ 楔积

$$(aw_1 + bw_2)\wedge w_3 = a\,w_1\wedge w_3 + b\,w_2\wedge w_3 \quad (p_1 = p_2)$$

$$(w_1\wedge w_2)\wedge w_3 = w_1\wedge(w_2\wedge w_3)$$

$$a(w_1\wedge w_2) = (aw_1)\wedge w_2$$

$$dx_i\wedge dx_j = -dx_j\wedge dx_i \Rightarrow dx_i\wedge dx_i = 0$$

$$a\,dx_1\wedge b\,dx_2 = -a(b\,dx_2\wedge dx_1) = -ab(dx_2\wedge dx_1)$$
$$= ab(dx_1\wedge dx_2)$$

$$\Sigma_p\, dx_{h_1}\wedge dx_{h_2}\wedge\cdots\wedge dx_{h_p} \quad 1\le h_1\le h_2\le\cdots\le h_p$$

Let d be the dimension of space

$p \le d$

Example 1.

$$w = (3dx + 4dy - dz) \wedge (dx - dy + 2dz)$$

$$= -3\,dx \wedge dy + 6\,dx \wedge dz + 4\,dy \wedge dx + 8\,dy \wedge dz$$

$$- dz \wedge dx + dz \wedge dy$$

$$= 7(dy \wedge dz - dz \wedge dx - dx \wedge dy)$$

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4. Complementary (Dual) Differential form

d, p-form, (d-p)-form   Hodge star operator

Metric / orientation

$*w$. 1. (indices of w) followed by (indices of w')

2. $(-1)^{n}$ (Metric tensor of diagonal element)

3D - Euclidean

$$*1 = dx_1 \wedge dx_2 \wedge dx_3$$

$$*dx_1 = dx_2 \wedge dx_3, \quad *dx_2 = dx_3 \wedge dx_1$$

$$*(dx_1 \wedge dx_2) = dx_3$$

$$*(dx_1 \wedge dx_2 \wedge dx_3) = 1$$

4D - Minkowski

$$*1 = dt \wedge dx \wedge dy \wedge dz$$

$$*(dt \wedge dx \wedge dy \wedge dz) = -1$$

$$* dx = dt \wedge dy \wedge dz$$

$$* dx_i = dt \wedge dx_j \wedge dx_k$$

$$*(dt \wedge dx) = -dy \wedge dz$$

$$*(dt \wedge dx_i) = -dx_j \wedge dx_k$$

$$A = A_x dx + A_y dy + A_z dz$$

$$B = B_x dx + B_y dy + B_z dz$$

$$C = C_x dx + C_y dy + C_z dz$$

$$A \wedge B = (A_y B_z - A_z B_y)\, dy \wedge dz + (A_x B_z - A_z B_x)\, dx \wedge dz$$

$$+ (A_x B_y - A_y B_x)\, dx \wedge dy$$

$$*(A \wedge B) = (\vec{A} \times \vec{B})_x\, dx + (\vec{A} \times \vec{B})_y\, dy + (\vec{A} \times \vec{B})_z\, dz$$

$$A \wedge B \wedge C = (A_x B_y C_z + \dots)\, dx \wedge dy \wedge dz$$

$$*(A \wedge B \wedge C) = \vec{A} \cdot (\vec{B} \times \vec{C})$$

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5. Exterior Derivative

$$d(w + w') = dw + dw' \qquad p = p'$$

$$d(fw) = (df) \wedge w + f\, dw$$

$$d(w \wedge w') = dw \wedge w' + (-1)^p\, w \wedge dw'$$

$$d(dw) = 0$$

$$df = \sum_j \frac{\partial f}{\partial x_j} dx_j$$

Example 2.

$$w = A\, dx_1 \wedge \cdots \wedge dx_p$$

$$w' = B\, dx_1 \wedge \cdots \wedge dx_{p'}$$

$$d(w \wedge w') = d(AB)\, [dx_1 \wedge \cdots \wedge dx_p] \wedge [dx_1 \wedge \cdots \wedge dx_{p'}]$$

$$= \left[\frac{\partial A}{\partial x^\mu} B + A \frac{\partial B}{\partial x^\mu}\right] dx_\mu \wedge [dx_1 \wedge \cdots \wedge dx_p] \wedge [dx_1 \wedge \cdots \wedge dx_{p'}]$$

$$= dw \wedge w' + (-1)^p w \wedge dw'$$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$d(df) = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} dx \wedge dy + \frac{\partial}{\partial x}\frac{\partial f}{\partial y} dy \wedge dx = 0$$

$$df = (\nabla f)_x dx + (\nabla f)_y dy + (\nabla f)_z dz$$

$$dw = d(A_x dx + A_y dy + A_z dz)$$

① $\Rightarrow$ $d(df) = \nabla \times \nabla f = 0$

$$= \left(\frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y}\right) dy \wedge dz + \cdots$$

$$= (\nabla \times A)_x\, dy \wedge dz + \cdots$$

$$\ast d(A_x dx + A_y dy + A_z dz) = (\nabla \times A)_x dx + \cdots$$

$$d(\underline{B_x dy\wedge dz + B_y dz \wedge dx + B_z dx\wedge dy}) \qquad ②$$

$$= \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right) dx\wedge dy \wedge dz$$

$$\ast d(B_x dy\wedge dz + B_y dz \wedge dx + B_z dx\wedge dy) = \nabla \cdot \vec{B}$$

令 $B = d(A_x dx + A_y dy + A_z dz)$

$$d(d(A_x dx + A_y dy + A_z dz)) = \nabla \cdot (\nabla \times A)\, dx\wedge dy\wedge dz = 0$$

Example 3.

$$F_{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix}$$

$$F = -E_x\, dt\wedge dx - E_y\, dt \wedge dy - E_z\, dt\wedge dz$$
$$+ B_x\, dy\wedge dz + B_y\, dz\wedge dx + B_z\, dx\wedge dy$$

$$J = \rho\, dx\wedge dy \wedge dz - J_x\, dt\wedge dy\wedge dz - J_y\, dt\wedge dz\, dx$$
$$- J_z\, dt \wedge dx\wedge dy$$

$$dF = 0$$

$\Rightarrow$

$$-\left[\frac{\partial E_x}{\partial y}dy + \frac{\partial E_x}{\partial z}dz\right]\wedge dt\wedge dx - \left[E_{y,x}dx + E_{y,z}dz\right]\wedge dt\wedge dy$$

$$-\left(E_{z,x}\overset{dx}{} + E_{z,y}\overset{dy}{}\right]\wedge dt\wedge dz + \left[B_{x,t}\overset{dt}{} + B_{x,x}\overset{dx}{}\right]\wedge dy\wedge dz +$$

$$\left(B_{y,t}dt + B_{y,y}dy\right]\wedge dz\wedge dx + \left(B_{z,t}dt + B_{z,z}dz\right)\wedge dx\wedge dy = 0$$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}, \qquad \nabla\cdot\vec{B} = 0$$

$$*F = E_x\, dy\wedge dz + E_y\, dz\wedge dx + E_z\, dx\wedge dy + B_x\, dt\wedge dx$$
$$+ B_y\, dt\wedge dy + B_z\, dt\wedge dz$$

$$d(*F) = (\nabla\cdot E)\, dx\wedge dy\wedge dz + \left[\frac{\partial E_x}{\partial t} - (\nabla\times B)_x\right] dt\wedge dy\wedge dz$$
$$+ \dots \quad = J$$

$$\nabla\cdot\vec{E} = \rho, \qquad \nabla\times\vec{B} - \frac{\partial\vec{E}}{\partial t} = \vec{J}$$

$$dJ = \left(\frac{\partial\rho}{\partial t} + \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}\right) dt\wedge dx\wedge dy\wedge dz = 0$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{J} = 0 \qquad \boxed{d(*F) = J}$$

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6. Integrating forms

1-form

$$\int_C w = \int_C (A_x dx + A_y dy) = \int_{t_p}^{t_q} \left(A_x(t)\frac{dx}{dt} + A_y(t)\frac{dy}{dt}\right] dt.$$

If it's independent about P and Q,

w is exact, $\quad w = df(x,y)$

若 $w = df$, $w = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$

$$\Rightarrow \int_P^Q w = f(Q) - f(P)$$

2 - form

$$\int_S w = \int_S B(x,y)\, dx \wedge dy$$

Let $x = au + bv$

$y = eu + fv$

$$dx \wedge dy = (a\,du + b\,dv) \wedge (e\,du + f\,dv)$$
$$= (af - be)\, du \wedge dv$$

Stoke's Theorem

Simply connected region R of p dimension differentiable manifold in n dimension space.

R has a boundary ∂R , which is $p-1$ d.

$w \rightarrow p-1$ form , dw.

then

$$\underline{\int_R dw = \int_{\partial R} w}$$

$$w = A\, dx_2 \wedge \cdots \wedge dx_p$$

$$\int_\Delta dw = \int_{\partial\Delta} \int_{x_1 - \delta}^{x} \frac{\partial A}{\partial x_1} dx_1 \wedge \cdots \wedge dx_p$$

$$= \int_{\partial\Delta} A(x_1, \cdots, x_p)\, dx_2 \wedge \cdots \wedge dx_p -$$

$$\int_{\partial\Delta} A(x_1^{-\delta}, \cdots, x_p)\, dx_2 \wedge \cdots \wedge dx_p$$

Example. 1. Green's Theorem in plane

$$w = P\,dx + Q\,dy$$

$$dw = \frac{\partial P}{\partial y} dy \wedge dx + \frac{\partial Q}{\partial x} dx \wedge dy = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy$$

$$\int_C P\,dx + Q\,dy = \int_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dS$$

2. $w = A_x\,dx + A_y\,dy + A_z\,dz$

$$dw = \frac{\partial A_x}{\partial y} dy \wedge dx + \frac{\partial A_x}{\partial z} dz \wedge dx + \cdots$$
$$\frac{\partial A_y}{\partial x} dx \wedge dy + \cdots + \cdots$$

$$\int_C A_x\,dx + A_y\,dy + A_z\,dz = \int_S (\nabla \times \vec{A}) \cdot d\vec{\sigma}$$

3. $w = E_x\,dy \wedge dz + E_y\,dz \wedge dx + E_z\,dx \wedge dy$

$$dw = \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) dx \wedge dy \wedge dz$$

$$\int_V (\nabla \cdot \vec{E})\, d\tau = \int_{\partial V} \vec{E} \cdot d\vec{\sigma}$$