m 为 V 的一个真子空间，则有另一个真子空间 m'，s.t. $V=m\oplus m'$

m' 为 m 的互补子空间

证明：$D=\dim V$，$m=\dim m$

$\therefore \quad 1\le m\le D-1$

$m \rightarrow m$ 个基底 $|v_1\rangle \cdots |v_m\rangle$

再在 V 中找 $|v_{m+1}\rangle \cdots |v_D\rangle$

$\forall |\psi\rangle \in V$,

$$\underset{V}{|\psi\rangle} = \underbrace{c_1|v_1\rangle+\cdots+c_m|v_m\rangle}_{m} + \underbrace{c_{m+1}|v_{m+1}\rangle+\cdots+c_D|v_D\rangle}_{m'}$$

$\therefore \quad V=m\oplus m'$，且 $\dim m' = \dim D - \dim m$.

$\forall |\psi\rangle \in V^D(\mathbb{F})$

$$|\psi\rangle = \alpha_1|v_1\rangle+\cdots+\alpha_D|v_D\rangle$$

$$= \underline{\alpha_1|v_1\rangle} + \cdots + \alpha_n|v_n\rangle + \cdots + \alpha_D|v_D\rangle$$

$$\Rightarrow \quad V^D(\mathbb{F}) = V_1^n(\mathbb{F}) \oplus V_2^{D-n}(\mathbb{F})$$

$$= m_1 \oplus m_2 \oplus \cdots \oplus m_D$$

投影算符：m 是 V 的一个子空间

iff $\forall |v\rangle \in V$，$\hat{P}|v\rangle = |u\rangle$

其中 $|v\rangle = |u\rangle + |u'\rangle$

$V = m\oplus m'$，$\hat{P}_{V\to m}$

可以证明：

$$\hat{P}(\alpha|v\rangle+\beta|w\rangle)$$

$$=\hat{P}(\alpha(|u\rangle+|u'\rangle)+\beta(|\eta\rangle+|\eta'\rangle))$$

$$=\hat{P}(\underbrace{\alpha|u\rangle+\beta|\eta\rangle}_{m}+\underbrace{\alpha|u'\rangle+\beta|\eta'\rangle}_{m'})$$

$$=\alpha\hat{P}|u\rangle+\beta\hat{P}|\eta\rangle$$

定理：$\hat{P}^2=\hat{P}$

证明：$\forall|v\rangle\in V$

$$V=m\oplus m'$$

$$\hat{P}^2|v\rangle=\hat{P}^2(|u\rangle+|u'\rangle$$

$$=\hat{P}\cdot(\hat{P}(|u\rangle+|u'\rangle))$$

$$=\hat{P}\cdot(|u\rangle+|0\rangle)$$

$$=|u\rangle=\hat{P}|v\rangle$$

$$\hat{1}=\underbrace{\hat{P}}_{\substack{\downarrow\\ m}}+\underbrace{\hat{1}-\hat{P}}_{\substack{\downarrow\\ m'}}$$

证明：$V=m\oplus m'$

有 $|v\rangle=|u\rangle+|u'\rangle$

$$(\hat{1}-\hat{P})|v\rangle=\hat{1}(|u\rangle+|u'\rangle)-\hat{P}(|u\rangle-|u'\rangle)$$

$$=|u\rangle+|u'\rangle-|u\rangle=|u'\rangle$$

① $\hat{1}: V\to V$

② $\hat{0}: V\to\{|0\rangle\}$

$\Rightarrow V=V\oplus\{|0\rangle\}$.

$\hat{A}$ 是 LO，有 $\hat{A}|v_1\rangle = \lambda_1|v_1\rangle$ $\quad |v_1\rangle \neq |v_2\rangle \neq 0$
$\hat{A}|v_2\rangle = \lambda_2|v_2\rangle$

① $\lambda_1 \neq \lambda_2$，则 $|v_1\rangle$ 和 $|v_2\rangle$ LI

$a_1|v_1\rangle + a_2|v_2\rangle = 0$

$|v_1\rangle = -\frac{a_2}{a_1}|v_2\rangle$

$-\frac{a_2}{a_1}\hat{A}|v_2\rangle = \lambda_1 \cdot (-\frac{a_2}{a_1})|v_2\rangle$

$\lambda_1 = \lambda_2$ ×

② $V_1 = \{\alpha_1|v_1\rangle \mid \alpha_1 \in \mathbb{F}\}$ 是 V 一子空间

③ V_1 在 $\hat{A}$ 作用下之不变子空间.

$\alpha_1\hat{A}|v_1\rangle = \alpha_1\lambda_1|v_1\rangle$

④ $D=2$，$V = V_1 \oplus V_2$.

$\underline{V = V_1 + V_2}$，$|w\rangle \in V_1 \cap V_2$

$|w\rangle = \alpha_1|v_1\rangle = \alpha_2|v_2\rangle$

$\alpha_1 = \alpha_2 = 0$，$\underline{|w\rangle = |0\rangle}$

$\Rightarrow V = V_1 \oplus V_2$

内积，$(|v\rangle, |w\rangle) \equiv \langle v|w\rangle \in \mathbb{F}$

且 ① $\langle w|v\rangle = \langle v|w\rangle^*$ $(\in \mathbb{C})$

② $\langle v|v\rangle \geqslant 0$, iff $|v\rangle = |0\rangle$

③ $\langle v|\alpha f+\beta g\rangle = \alpha\langle v|f\rangle + \beta\langle v|g\rangle$

$\langle \alpha f+\beta g|v\rangle = \alpha^*\langle f|v\rangle + \beta\langle g|v\rangle$

$\langle v|w\rangle = (|v\rangle, |w\rangle)$

$\langle v| = (|v\rangle, \_\_)$

$\langle v|\hat{A} = (|v\rangle, \hat{A}\_\_)$

$|v\rangle\langle w| = |v\rangle(|w\rangle, \_\_)$

性质: 1. $\langle v|0\rangle = 0$

2. $\langle v|f\rangle = 0$, $\forall |v\rangle$, $|f\rangle = 0$

3. iff $\langle v|f\rangle = \langle v|g\rangle$, $\forall |v\rangle$

$|f\rangle = |g\rangle$.

定理: $\alpha_1|v_1\rangle + \cdots + \alpha_n|v_n\rangle = 0$

$\alpha_1\langle v_1|v_1\rangle = 0 \Rightarrow \alpha_1 = 0$

Gram - Schmidt 正交化 .

$|\psi\rangle = \langle e_1|\psi\rangle|e_1\rangle + \cdots + \langle e_n|\psi\rangle|e_n\rangle$

$= (|e_1\rangle\langle e_1| + \cdots + |e_n\rangle\langle e_n|)|\psi\rangle = \hat{I}|\psi\rangle$

$$\Rightarrow \quad |e_1\rangle\langle e_1| + \cdots + |e_n\rangle\langle e_n| = \hat{1}$$

$|v_1\rangle$ --- $|v_n\rangle$ $\qquad \psi = c_1|v_1\rangle + \cdots + c_n|v_n\rangle$

$$\begin{pmatrix} \langle v_1|\psi\rangle \\ \langle v_2|\psi\rangle \\ \vdots \\ \langle v_n|\psi\rangle \end{pmatrix} = \begin{pmatrix} \langle v_1|v_1\rangle & \langle v_1|v_2\rangle & \cdots & \langle v_1|v_n\rangle \\ \langle v_2|v_1\rangle & \langle v_2|v_2\rangle & \cdots & \cdots \\ & \ddots & \ddots & \\ & & & \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

$$c_i = V^{-1}_{ij}\langle v_j|\psi\rangle$$

$$|\psi\rangle = c_i|v_i\rangle = V^{-1}_{ij}\langle v_i|\psi\rangle|v_i\rangle = \left(V^{-1}_{ij}|v_i\rangle\langle v_j|\right)|\psi\rangle$$

$$\Rightarrow \quad V^{-1}_{ij}|v_i\rangle\langle v_j| = \hat{1}$$

三个不等式.

Schwarz: $\forall |v\rangle, |w\rangle$

$$|\langle v|w\rangle| \le \|v\|\,\|w\| \qquad \text{“=” 线性相关}$$

证: $|w\rangle = 0$, 成立

$|w\rangle \ne 0$

令. $|z\rangle = \left(|v\rangle - \dfrac{\langle w|v\rangle}{\|w\|^2}|w\rangle\right)$

$$\langle z|z\rangle = \langle v|v\rangle + \frac{|\langle w|v\rangle|^2}{\|w\|^2} - \frac{\langle w|v\rangle}{\|w\|^2}\langle v|w\rangle - \frac{\langle w|v\rangle}{\|w\|^2}\langle w|v\rangle = \|v\|^2 - \frac{|\langle w|v\rangle|^2}{\|w\|^2} = \|z\|^2 \ge 0$$

$\Rightarrow \langle w|v\rangle \leq \|v\|\,\|w\|$

三角不等式. $\forall |v\rangle, |w\rangle$ $\quad$ Im=0, Re>0.

$\|v+w\| \leq \|v\| + \|w\|$ $\quad$ "=" 线性相关

$$\langle v+w|v+w\rangle = \langle v|v\rangle + \langle w|w\rangle + \langle w|v\rangle + \langle v|w\rangle$$

$$= \|v\|^2 + \|w\|^2 + 2\mathrm{Re}\langle v|w\rangle$$

$$\leq \|v\|^2 + \|w\|^2 + 2\left|\langle v|w\rangle\right|$$

$$\leq (\|w\| + \|v\|)^2 \quad ✓$$

Bessel 不等式 $|e_1\rangle \cdots |e_m\rangle$ 正交归一.

$$\|v\|^2 \geq \sum_{i=1}^{m} |c_i|^2, \text{ 其中 } c_i = \langle e_i|v\rangle$$

且 $|v'\rangle = |v\rangle - \sum_{i=1}^{m} c_i|e_i\rangle$ 与 $|e_1\rangle \cdots |e_m\rangle$ 正交.

证: $\langle v'|v'\rangle = \langle v|v\rangle + c_i^* c_j \langle e_i|e_j\rangle - c_i\langle v|e_i\rangle - c_i^*\langle e_i|v\rangle$

$$= \|v\|^2 + c_i^* c_i - c_i c_i^* - c_i^* c_i$$

$$= \|v\|^2 - \sum_i |c_i|^2 = \|v'\|^2 \geq 0$$

$\langle e_i|v'\rangle$

$= \langle e_i|v - c_j e_j\rangle$

$= \langle e_i|v\rangle - c_j\langle e_i|e_j\rangle$

$= \langle e_i|v\rangle - c_i = 0$

性质:

① $\langle v|\hat{0}|w\rangle = 0$

② $\forall |v\rangle, |w\rangle$

$\langle v|\hat{A}|w\rangle = 0$, $\hat{A} = \hat{0}$

③ $\langle v|\hat{A}|w\rangle = \langle v|\hat{B}|w\rangle$

$\Rightarrow \quad \hat{A} = \hat{B}$

$\forall |v\rangle, |w\rangle,$

$$\langle w | \hat{B} | v \rangle = \langle v | \hat{A} | w \rangle^*$$

$\hat{B}$ 是 $\hat{A}$ 的厄米共轭， $B = \hat{A}^\dagger$， $\dagger$ dagger

$$\Rightarrow \quad \langle w | \hat{A}^\dagger | v \rangle = \langle v | \hat{A} | w \rangle^*$$

$$= \left(|w\rangle, \hat{A}^\dagger |v\rangle \right)$$

$$= \left(\hat{A}^\dagger |v\rangle, |w\rangle \right)^*$$

$$\Rightarrow \quad \langle \hat{A}^\dagger v | w \rangle = \langle v | \hat{A} | w \rangle$$

$$(\hat{A}^\dagger)^\dagger = \hat{A}$$

证:
$$\langle v | \hat{A} | w \rangle$$
$$= \langle w | \hat{A}^\dagger | v \rangle^*$$
$$= \langle (A^\dagger)^\dagger w | v \rangle^*$$
$$= \langle v | (\hat{A}^\dagger)^\dagger | w \rangle$$

$$(\hat{A}\hat{B})^\dagger = B^\dagger A^\dagger$$

$$(\hat{A} + \hat{B})^\dagger = \hat{A}^\dagger + \hat{B}^\dagger$$

$$(|f\rangle\langle g|)^\dagger = |g\rangle\langle f|$$

$\hat{A}^\dagger$ 也为 LO

$$\langle v | \hat{A}^\dagger (\alpha f + \beta g) \rangle$$
$$= \langle \alpha f + \beta g | \hat{A} | v \rangle^*$$
$$= \langle \hat{A} v | \alpha f + \beta g \rangle$$
$$= \alpha \langle \hat{A} v | f \rangle + \beta \langle \hat{A} v | g \rangle$$
$$= \alpha \langle v | A^\dagger | f \rangle + \beta \langle v | \hat{A}^\dagger | g \rangle$$

线性厄米算符

$\hat{A}^\dagger = \hat{A}$

① $\langle v|\hat{A}|v\rangle \in \mathbb{R}$.

$= \langle v|\hat{A}^\dagger|v\rangle^* = \langle v|\hat{A}|v\rangle^*$

② $(\hat{A}+\hat{B})^\dagger$

$= \hat{A}^\dagger + \hat{B}^\dagger = \hat{A}+\hat{B}$

③ 若 $[\hat{A},\hat{B}]=0$, $\hat{A}\cdot\hat{B}$

$(\hat{A}\cdot\hat{B})^\dagger = \hat{B}^\dagger\cdot\hat{A}^\dagger = \hat{B}\cdot\hat{A} = \hat{A}\cdot\hat{B}$

线性幺正算符.

$\hat{U}^\dagger = \hat{U}^{-1}$

① $(\hat{U}_1\cdot\hat{U}_2)^\dagger$

$= \hat{U}_2^\dagger U_1^\dagger = U_2^{-1}U_1^{-1} = (\hat{U}_1\hat{U}_2)^{-1}$

② $\langle \hat{U}f|\hat{U}g\rangle$

$= \langle f|\hat{U}^\dagger\hat{U}g\rangle$

$= \langle f|g\rangle$

③ $U^\dagger$ 亦为幺正

内积空间的正交互补性

$m \perp m^\perp$

[定理] m 是 V 的一子空间,

$m^\perp = \{|v\rangle \mid |v\rangle \in V$ 且 $\langle v|v'\rangle = 0, |v'\rangle \in m\}$

则 ① $m^\perp$ 为 V 一个子空间

② $m \oplus m^\perp = V$ (正交直和)

证: ② 令 $|v\rangle \in m \cap m^\perp$

$\therefore \langle v|v\rangle = 0 \Rightarrow \|V\| = 0 \Rightarrow |v\rangle = |0\rangle$

$\forall |\varphi\rangle \in V$, 令 $|e_1\rangle \cdots |e_m\rangle$ 是 m 中正交归一之基底

令 $|f\rangle = \alpha_i|e_i\rangle$, $\alpha_i = \langle e_i|\varphi\rangle$

令 $|g\rangle = |\varphi\rangle - |f\rangle = |\varphi\rangle - \alpha_i|e_i\rangle$

$$\langle g|f\rangle = \langle \psi|f\rangle - \langle f|f\rangle$$
$$= \alpha_i \langle \psi|e_i\rangle - \alpha_j^* \alpha_i \langle e_j|e_i\rangle$$
$$= \alpha_i^* \alpha_i - \alpha_i^* \alpha_i = 0$$

故 $|g\rangle \in M^\perp$

$$|\psi\rangle = |f\rangle + |g\rangle$$

$$V = M \oplus M^\perp = M_1' \oplus M_2' \oplus \cdots + M_0'$$

投影算符

若 $\hat{P}^2 = \hat{P} = \hat{P}^\dagger$，$\hat{P}$ 是投影.

$\Leftarrow$ $V = M \oplus M'$

$$\langle v|\hat{P}|v\rangle = \langle v|u\rangle = \langle u+u'|u\rangle$$
$$= \langle u|u\rangle + \underline{\langle u'|u\rangle} = \|u\|^2 \in \mathbb{R}$$
$$\langle v|\hat{P}^\dagger|v\rangle = \langle \hat{P}v|v\rangle = \langle v|\hat{P}|v\rangle^* = \langle v|\hat{P}|v\rangle$$

$$\Rightarrow \quad \hat{P}^\dagger = \hat{P}$$

$\Rightarrow$ $M = \{|u\rangle \mid \hat{P}|u\rangle = |u\rangle,\ |u\rangle \in V\}$

M 是 V 的一个子空间.

$V = M \oplus M^\perp$，其中 $M^\perp \perp M$.

$\forall\ |v\rangle \in V$

$$|v\rangle = \underset{M}{|u\rangle} + \underset{M_\perp}{|u_\perp\rangle}$$

有 $\hat{P}|v\rangle = \hat{P}(|u\rangle + |u_\perp\rangle) = \hat{P}|u\rangle + \underline{\hat{P}|u_\perp\rangle} = \hat{P}|u\rangle = |u\rangle$

$\because \underline{\langle \hat{P}u_\perp}|\hat{P}u_\perp\rangle = \langle u_\perp|\hat{P}^\dagger \hat{P} u_\perp\rangle = \langle u_\perp|\hat{P}^2|u_\perp\rangle = \langle u_\perp|\hat{P}|u_\perp\rangle$

$= 0$

$\because \hat{P}^2|u_\perp\rangle = \underline{\hat{P}|u_\perp\rangle}$

$\Rightarrow \hat{P}|u_\perp\rangle \in m$

$\langle u_\perp|\hat{P}u_\perp\rangle = 0$

$\therefore \hat{P}$ 是 $V \to m$ 投影

$\forall |f\rangle, |g\rangle$

$\langle f|\hat{A}|g\rangle = \langle f|\hat{B}|g\rangle \Rightarrow \hat{A} = \hat{B}$

$\Updownarrow$

是否.

$\forall |w\rangle$

$\langle w|\hat{A}|w\rangle = \langle w|\hat{B}|w\rangle \Rightarrow \hat{A} = \hat{B}$

$\text{令}\quad |\psi\rangle = |\psi_1\rangle + |\psi_2\rangle$

$|\psi'\rangle = |\psi_1\rangle + i|\psi_2\rangle$

$\langle\psi|\hat{A}|\psi\rangle =$

$= \langle\psi_1+\psi_2|\hat{A}|\psi_1+\psi_2\rangle$

$= \cancel{\langle\psi_1|\hat{A}|\psi_1\rangle} + \cancel{\langle\psi_2|\hat{A}|\psi_2\rangle} = \cancel{\langle\psi_1|\hat{B}|\psi_1\rangle} + \cancel{\langle\psi_2|\hat{B}|\psi_2\rangle}$

$+ \langle\psi_1|\hat{A}|\psi_2\rangle + \langle\psi_2|\hat{A}|\psi_1\rangle \quad + \langle\psi_1|\hat{B}|\psi_2\rangle + \langle\psi_2|\hat{B}|\psi_1\rangle$

$\Rightarrow \langle\psi_1|\hat{A}|\psi_2\rangle + \langle\psi_2|\hat{A}|\psi_1\rangle = \langle\psi_1|\hat{B}|\psi_2\rangle + \langle\psi_2|\hat{B}|\psi_1\rangle$

$$\langle\psi'|\hat{A}|\psi'\rangle$$

$$= \cancel{\langle\psi_1|\hat{A}|\psi_1\rangle} + \cancel{\langle\psi_2|\hat{A}|\psi_2\rangle} = \cancel{\langle\psi_1|\hat{B}|\psi_1\rangle} + \cancel{\langle\psi_2|\hat{B}|\psi_2\rangle}$$

$$+ i\langle\psi_1|\hat{A}|\psi_2\rangle - i\langle\psi_2|\hat{A}|\psi_1\rangle + i\langle\psi_1|\hat{B}|\psi_2\rangle - i\langle\psi_2|\hat{B}|\psi_1\rangle$$

$$\Rightarrow \quad \langle\psi_1|\hat{A}|\psi_2\rangle - \langle\psi_2|\hat{A}|\psi_1\rangle = \langle\psi_1|\hat{B}|\psi_2\rangle - \langle\psi_2|\hat{B}|\psi_1\rangle$$

$$\Rightarrow \quad \langle\psi_1|\hat{A}|\psi_2\rangle = \langle\psi_1|\hat{B}|\psi_2\rangle$$

$\hat{P}\hat{P}_2 = \hat{0}$，则相互正交

$$(\hat{P}_1\hat{P}_2)^\dagger = \hat{0}^\dagger = \hat{0}$$

定理：设 $\hat{P}_1$、$\hat{P}_2$ 相互正交，$\hat{P}_1 : V \to m_1$，$\hat{P}_2 : V \to m_2$，

$m_1 \perp m_2$

证：$\begin{cases} \hat{P}_1|v_1\rangle = |v_1\rangle \\ \hat{P}_2|v_2\rangle = |v_2\rangle \end{cases}$

$\Rightarrow$

$$\langle v_1|v_2\rangle = \langle\hat{P}_1 v_1|\hat{P}_2 v_2\rangle = \langle v_1|\hat{P}_1^\dagger\hat{P}_2|v_2\rangle$$
$$= \langle v_1|\hat{P}_1\hat{P}_2|v_2\rangle$$
$$= 0$$

$\Leftarrow$ $m_1 \perp m_2$，$\forall |v\rangle \in V$

$$|v\rangle = |v_1\rangle + |v_1'\rangle = |v_2\rangle + |v_2'\rangle$$

且 $\hat{P}_1|v\rangle = |v_1\rangle$，$\hat{P}_2|v\rangle = |v_2\rangle$

$\therefore \quad \langle v_1 | v_2 \rangle = 0$

$$\langle \hat{P}_1 v_1 | \hat{P}_2 v_2 \rangle = \langle v | \hat{P}_1^\dagger \hat{P}_2 | v \rangle = \langle v_1 | \hat{P}_1 \hat{P}_2 | v_2 \rangle = 0$$

$\Rightarrow \quad \hat{P}_1 \hat{P}_2 = \hat{0}$

$V = m \oplus m'$

$\hat{P} : V \to m, \quad \hat{1} - \hat{P} : V \to m'$

$|\psi\rangle = |v\rangle + |v'\rangle$

$$\begin{aligned} \hat{1} |\psi\rangle &= (\hat{P} + \hat{1} - \hat{P}) |\psi\rangle \\ &= \hat{P} |\psi\rangle + (\hat{1} - \hat{P}) |\psi\rangle \\ &= |v\rangle + |v'\rangle \end{aligned}$$

$(\hat{1} - \hat{P})^2 = (\hat{1} - \hat{P}) = (\hat{1} - \hat{P})^\dagger$

$(\hat{1} - \hat{P}) \cdot \hat{P} = \hat{P} - \hat{P}^2 = 0$

故 $\quad m \perp m'$

$\hat{P}_1, \hat{P}_2$ 均投影， iff $\hat{P}_1, \hat{P}_2$ 正交， $\hat{P}_1 + \hat{P}_2$ 也是投影

$\Rightarrow \quad \hat{P}_1 \cdot \hat{P}_2 = \hat{P}_2 \cdot \hat{P}_1 = 0$

$$\begin{aligned} (\hat{P}_1 + \hat{P}_2)^2 &= \hat{P}_1^2 + \hat{P}_2^2 + \hat{P}_1 \cdot \hat{P}_2 + \hat{P}_2 \cdot \hat{P}_1 \\ &= \hat{P}_1 + \hat{P}_2 \end{aligned}$$

$(\hat{P}_1 + \hat{P}_2)^\dagger = \hat{P}_1 + \hat{P}_2$

$\Leftarrow \quad (\hat{P}_1+\hat{P}_2)^2=\hat{P}_1+\hat{P}_2 \quad \Rightarrow \quad \hat{P}_1\hat{P}_2+\hat{P}_2\hat{P}_1=0$

$=\hat{P}_1+\hat{P}_1\hat{P}_2+\hat{P}_2\hat{P}_1+P_2$

① $\hat{P}_1(\hat{P}_1\hat{P}_2+\hat{P}_2\hat{P}_1)=\hat{P}_1\cdot\hat{0}=\hat{0}$

$=\hat{P}_1P_2+\hat{P}_1\hat{P}_2\hat{P}_1=\hat{0}$

$\Rightarrow \quad \hat{P}_1\hat{P}_2=-\hat{P}_1\hat{P}_2\hat{P}_1$

② $(\hat{P}_1\hat{P}_2+\hat{P}_2\hat{P}_1)\hat{P}_1=\hat{0}\cdot\hat{P}_1=\hat{0} \quad \Rightarrow \quad \hat{P}_1\hat{P}_2=\hat{P}_2\hat{P}_1=\hat{0}$

$\hat{P}_1\hat{P}_2\hat{P}_1+\hat{P}_2\hat{P}_1=\hat{0}$

$\Rightarrow \quad \hat{P}_2\hat{P}_1=-\hat{P}_1\hat{P}_2\hat{P}_1$

$\hat{P}_1\cdots\hat{P}_n$ 投影， iff $\hat{P}_1\cdots\hat{P}_n$ 两两正交，则 $P_1+\cdots+P_n$ 投影

证：

$\Rightarrow \quad (P_1+\cdots+P_n)^2=\sum_{ij}\hat{P}_i\hat{P}_j=P_iP_i=P_1+\cdots+P_n$

$(\text{———})^\dagger=\text{———}$

$\Leftarrow \quad \hat{P}_i^2=\hat{P}_i=\hat{P}_i^\dagger$

用 Bessel 不等式，

$\forall |v\rangle\in V$，有 $\|v\|^2=\langle v|v\rangle\geq\langle\hat{P}v|\hat{P}v\rangle$

$=\langle v|\hat{P}^\dagger\hat{P}|v\rangle=\langle v|\hat{P}|v\rangle$

$=\langle v|(\hat{P}_1+\cdots+\hat{P}_n)|v\rangle=\sum_{i=1}^{n}\langle v|\hat{P}_i|v\rangle=\sum_{i=1}^{n}\langle v|\hat{P}_i^\dagger\hat{P}_i|v\rangle$

$=\sum_{i=1}^{n}\langle\hat{P}_iv|\hat{P}_iv\rangle$

设 $|v\rangle = \hat{P}_k|\psi\rangle$，有 $\cancel{\langle \hat{P}_k\psi|\hat{P}_k\psi\rangle} \geq \sum_i \langle \hat{P}_i\hat{P}_k\psi|\hat{P}_i\hat{P}_k\psi\rangle$

$$= \cancel{\langle \hat{P}_k^2\psi|\hat{P}_k^2\psi\rangle} + \sum_{i\neq k}\langle \hat{P}_i\hat{P}_k\psi|\hat{P}_i\hat{P}_k\psi\rangle$$

$$\Rightarrow \sum_{i\neq k}\langle \hat{P}_i\hat{P}_k\psi|\hat{P}_i\hat{P}_k\psi\rangle \leq 0$$

$$\Rightarrow \hat{P}_i\hat{P}_k|\psi\rangle = 0$$

$$\Rightarrow \hat{P}_i\hat{P}_k = \hat{0}$$

$$|e_1\rangle\langle e_1| + \cdots + |e_D\rangle\langle e_D| = \hat{1}$$

$$V^D(\mathbb{F}) = V_1'(\mathbb{F}) \oplus V_2'(\mathbb{F}) \oplus \cdots \oplus V_D'(\mathbb{F})$$

$$\begin{cases} |\psi\rangle = c_1|e_1\rangle + \cdots + c_D|e_D\rangle \\ \hat{1} = \hat{P}_1 + \cdots + \hat{P}_D \end{cases}$$

表示：相对一组基底

$$|\psi\rangle \xrightarrow{\{|v_i\rangle\}} \begin{pmatrix} c_1 \\ \vdots \\ c_D \end{pmatrix}$$

$$\hat{A} \longrightarrow \begin{pmatrix} A_{11} & \cdots & A_{D1} \\ & \ddots & \\ A_{1D} & \cdots & A_{DD} \end{pmatrix}$$

$$\begin{cases} \hat{A}|v_1\rangle = A_{11}|v_1\rangle + \cdots + A_{D1}|v_D\rangle \\ \vdots \\ \hat{A}|v_D\rangle = A_{1D}|v_D\rangle + \cdots + A_{DD}|v_D\rangle \end{cases}$$

$$\hat{A}\hat{B}|f\rangle = |g\rangle \Rightarrow \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix} = \begin{pmatrix} & & \\ & A_B & \\ & & \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

线性算符.

$\hat{A}$ 是 LO，iff λ 是 $\hat{A}$ 的本征值，则 $\hat{A}-\lambda\hat{1}$ 是不可逆.

证：$\Leftarrow$ iff $\hat{T}|v\rangle=|0\rangle$，唯一解 $|v\rangle=|0\rangle$，$\hat{T}$ 可逆

$$(\hat{A}-\lambda\hat{1})|v\rangle = |0\rangle$$

$$\Downarrow$$

$$\hat{A}|v\rangle = \lambda|v\rangle, \quad |v\rangle \neq 0$$

$\Rightarrow$ $\forall |v\rangle \neq 0$

$$\hat{A}|v\rangle = \lambda|v\rangle$$

$$(\hat{A}-\lambda\hat{1})|v\rangle = 0 \checkmark$$

线性厄米算符的本征值定理.

[定理一] ① $\hat{A}$ 的本征值 $\in \mathbb{R}$

② 不同 λ 的 $|v_i\rangle$ 正交

证：① $\hat{A}|v\rangle = \lambda|v\rangle$

左 $= \langle v|\hat{A}|v\rangle = \lambda\langle v|v\rangle$

右 $= \langle \hat{A}v|v\rangle = \langle \lambda v|v\rangle = \lambda^*\langle v|v\rangle \Rightarrow \lambda = \lambda^*$

$= \langle v|\hat{A}^\dagger|v\rangle = \langle v|\hat{A}|v\rangle$ $\quad \lambda \in \mathbb{R}$.

② $\hat{A}|v_1\rangle=\lambda_1|v_1\rangle \Rightarrow \langle v_2|\hat{A}|v_1\rangle=\lambda_1\langle v_2|v_1\rangle$

$\hat{A}|v_2\rangle=\lambda_2|v_2\rangle \Rightarrow \langle\hat{A}v_2|v_1\rangle=\lambda_2^*\langle v_2|v_1\rangle=\lambda_2\langle v_2|v_1\rangle$

$=\langle v_2|\hat{A}^{+}v_1\rangle=\langle v_2|\hat{A}|v_1\rangle=\lambda_1\langle v_2|v_1\rangle$

$\Rightarrow \langle v_2|v_1\rangle=0$

[定理二] (简并)

$V^{p}(\mathbb{F})$ 至少有一组基底由 $\hat{A}$ 的本征向量 (D取无限)

证： $\hat{A}-\lambda_1\hat{I}$，故 $\exists\ |v_1\rangle\neq 0$

s.t. $(\hat{A}-\lambda_1\hat{I})\underline{|v_1\rangle}=0$

令 $m_1=\{\alpha_1|v_1\rangle\ |\ \alpha_1\in\mathbb{F}\}$

$V^{D}=m_1\oplus m_1^{\perp}$

对于 $\forall\ |u\rangle\in m_1^{\perp}$， $\langle v_1|\hat{A}|u\rangle=(\hat{A}^{+}v_1|u\rangle=\langle\lambda_1 v_1|u\rangle=0$

$\therefore$ $m^{\perp}$ 是 $\hat{A}$ 作用下一不变子空间

$\therefore$ $|v_2\rangle$ - - -

$\forall\ \hat{A}$ $\quad \hat{A}=\hat{I}\cdot\hat{A}\cdot\hat{I}$

$=|e_i\rangle\langle e_i|\hat{A}|e_j\rangle\langle e_j|$

$=A_{ij}|e_i\rangle\langle e_j|$

厄米 算符 $\Rightarrow$ 厄米
幺正 $\quad\quad\quad$ 幺正 矩阵

任一N阶厄米阵一定可对角化

$$\hat{A} \longrightarrow \begin{matrix} \lambda_1 \cdots \lambda_N \\ |v_1\rangle \cdots |v_N\rangle \end{matrix} \leftarrow \text{正交}$$

$$A \longrightarrow A' = S^{-1} A S = S^{\dagger} A S$$

其中 $S = \left(|v_1\rangle, |v_2\rangle, \cdots, |v_N\rangle \right)$

$$AS = \left(\langle e_i | A | e_j \rangle \right) \left(|v_1\rangle, \cdots, |v_N\rangle \right) = \left(\lambda_1 |v_1\rangle \cdots \lambda_N |v_N\rangle \right)$$

$$S^{\dagger} A S = \begin{pmatrix} |v_1\rangle^* \\ \vdots \\ |v_N\rangle^* \end{pmatrix} \left(\lambda_1 |v_1\rangle, \cdots, \lambda_N |v_N\rangle \right) = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix}$$

---

iff $[\hat{A}, \hat{B}] = 0$，则 $\hat{A}, \hat{B}$ 至少有一组共同的本征向量组.

证： $\hat{A} |v_i\rangle = a_i |v_i\rangle$

$\Leftarrow$ $\hat{B} |v_i\rangle = b_i |v_i\rangle$

$$\forall |\psi\rangle, \quad \hat{A}\hat{B}|\psi\rangle = \hat{A}(\hat{B}|\psi\rangle) = \hat{A}\left(\hat{B} \sum_i c_i |v_i\rangle\right)$$

$$= \hat{A} \sum_i b_i c_i |v_i\rangle$$

$$= \sum_i a_i b_i c_i |v_i\rangle = \hat{B}\hat{A}|\psi\rangle$$

$\Rightarrow \hat{A}\hat{B} = \hat{B}\hat{A}$

$\Rightarrow$ $\hat{A}\hat{B} = \hat{B}\hat{A}$, $|v\rangle \neq 0$

$\hat{A}|v\rangle = \lambda |v\rangle$

$\because \hat{A}(\hat{B}|v\rangle) = \hat{A}\hat{B}|v\rangle = \hat{B}(\lambda|v\rangle) = \lambda \hat{B}|v\rangle$

① 无简并

$\hat{B}|v\rangle$ 与 $|v\rangle$ 同向.

$\hat{B}|v\rangle = \delta|v\rangle$，有相同本征向量组.

② 有简并，

$$\begin{cases} \hat{A}|v_1\rangle = \lambda|v_1\rangle \\ \hat{A}|v_2\rangle = \lambda|v_2\rangle \end{cases}$$

可得 $B|v_1\rangle = c_{11}|v_1\rangle + c_{21}|v_2\rangle$

$B|v_2\rangle = c_{12}|v_1\rangle + c_{22}|v_2\rangle$

$$\begin{pmatrix} c_{11} & c_{21} \\ c_{12} & c_{22} \end{pmatrix} = \begin{pmatrix} \langle v_1|\hat{B}|v_1\rangle & \langle v_1|\hat{B}|v_2\rangle \\ \langle v_2|\hat{B}|v_1\rangle & \langle v_2|\hat{B}|v_2\rangle \end{pmatrix} \leftarrow \text{厄米阵}$$

$\downarrow$

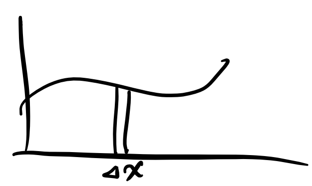
可被对角化 $\begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}$

$|v_1\rangle, |v_2\rangle \longrightarrow |u_1\rangle, |u_2\rangle$

$$\begin{cases} \hat{B}|u_1\rangle = b_1|u_1\rangle \\ \hat{B}|u_2\rangle = b_2|u_2\rangle \end{cases}$$

$\hat{A}, \hat{B}$ 有共同本征向量. 证完

基底 $|x_n\rangle$



$|f_n\rangle = \langle x_i|f_n\rangle|x_i\rangle = f(x_i)|x_i\rangle$

内积 $\langle f_N|g_N\rangle = \langle f_N|x_i\rangle\langle x_i|g_N\rangle = \sum_i f^*(x_i)\,g(x_i)$

归一正交 $\longrightarrow$

$\{|x_i\rangle\}$ $\qquad$ $\{|\hat{x}_i\rangle\}$，$|\hat{x}_i\rangle = \dfrac{|x_i\rangle}{\sqrt{\Delta x}}$

$\langle x_i|x_j\rangle = \delta_{ij}$ $\qquad$ $\langle \hat{x}_i|\hat{x}_j\rangle = \dfrac{\delta_{ij}}{\Delta x}$

$|x_i\rangle\langle x_i| = \hat{1}$ $\qquad$ $\Delta x|\hat{x}_i\rangle\langle\hat{x}_j| = \cancel{\hat{1}}$

$$|\tilde{f}_N\rangle = \sqrt{\Delta x}\,|f_N\rangle$$

$$f(x_i) = \langle x_i | f_N\rangle = \langle \hat{x}_i | \tilde{f}_N\rangle = f(x_i)$$

$$\langle \tilde{f}_N | \tilde{g}_N\rangle = \sum_i \Delta x\, f^*(x_i)\, g(x_i) = \int_a^b f(x)^* g(x)\, dx$$

$$\| f_N \|^2 = \int_a^b dx\, |f(x)|^2$$

$\xrightarrow{\text{取 } N\to+\infty}$ 归 $\sqrt{\delta(0)}$ 正交. $\langle x | x'\rangle = \delta(x-x')$

$$\int dx\, |x\rangle\langle x'| = \hat{1}$$

$$|f\rangle \xrightarrow{\{|x\rangle\}} \langle x|f\rangle = f(x)$$

$$|f\rangle = \int dx\, f(x)\, |x\rangle$$

例：$|\psi\rangle = \hat{1}\cdot|\psi\rangle = \int dx\, |x\rangle\langle x|\psi\rangle = \int dx\, \psi(x)\, |x\rangle$

$$\langle f|g\rangle = \langle f|\hat{1}|g\rangle = \langle f| \int dx\, |x\rangle\langle x|g\rangle$$
$$= \int dx\, \langle f|x\rangle\langle x|g\rangle = \int dx\, f^*(x)\, g(x)$$

微分算符.

$$\langle x|\hat{D}|f\rangle = \frac{d}{dx} f(x) = \frac{d}{dx}\langle x|f\rangle$$

$$\langle x|\hat{D}|x'\rangle = \frac{d}{dx}\langle x|x'\rangle = \frac{d}{dx}\delta(x-x')$$

$$\langle f|\hat{D}|g\rangle = \langle f|\hat{1}\cdot\hat{D}|g\rangle$$
$$= \int dx\, \langle f|x\rangle\langle x|\hat{D}|g\rangle = \int dx\, f^*(x)\, \frac{d}{dx} g(x)$$

$$= f^*(x)g(x) - \int dx \frac{df^*(x)}{dx} g(x)$$

$$\langle f|\hat{D}|g\rangle = -\langle g|\hat{D}|f\rangle^* = -\langle f|\hat{D}^\dagger|g\rangle \qquad (反厄米)$$

可定义 $\hat{K} = -i\hat{D}$ 为厄米

令 $\hat{K}|k\rangle = k|k\rangle$, $k \in \mathbb{R}$

$$\langle x|\hat{K}|k\rangle = \langle x|k|k\rangle = k\langle x|k\rangle$$

$$\begin{aligned} &\parallel \\ &-i\langle x|\hat{D}|k\rangle \\ &= -i\frac{d}{dx}\underbrace{\langle x|k\rangle}_{\phi_k(x)} \end{aligned}$$

$$\Rightarrow -i\frac{d\phi_k(x)}{dx} = k\phi_k(x)$$

$$\Rightarrow \phi_k(x) = Ae^{ikx}, \quad k \in \mathbb{R}.$$

$$\begin{aligned} \langle k|k'\rangle &= \langle k|\hat{1}|k'\rangle \\ &= \int_{-\infty}^{+\infty} dx \langle k|x\rangle\langle x|k'\rangle \\ &= A^*A'\int_{-\infty}^{+\infty} dx\, e^{i(k'-k)x} \\ &= 2\pi A^*A'\delta(k-k') \\ &\underline{\underline{k=k'}}\ 2\pi|A|^2\delta(0) \end{aligned}$$

令 $A = \frac{1}{\sqrt{2\pi}}$, $\langle k|k'\rangle = \delta(k-k')$

$$|\psi\rangle = \int_{-\infty}^{+\infty} dk\, \tilde{\psi}(k)|k\rangle$$

$$\Rightarrow \langle k' | \psi \rangle = \int_{-\infty}^{+\infty} \tilde{\varphi}(k) \langle k | k' \rangle = \tilde{\varphi}(k')$$

$$\Rightarrow \quad |\psi\rangle = \int_{-\infty}^{+\infty} dk \, \langle k | \psi \rangle |k\rangle = \left(\int_{-\infty}^{+\infty} dk \, |k\rangle\langle k| \right) |\psi\rangle$$

$$\int_{-\infty}^{+\infty} dk \, |k\rangle\langle k| = \hat{1}$$

$$\hat{A} = \hat{1} \cdot \hat{A} \cdot \hat{1} = \int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} |x'\rangle\langle x'| \hat{A} |x\rangle\langle x| \, dx$$

$$= \int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} dx \quad A(x, x') \, |x'\rangle\langle x|$$

$$\hat{X} \quad , \quad \hat{k} \longrightarrow |\psi\rangle = \hat{1} |\psi\rangle = \int_{-\infty}^{+\infty} dk \, \varphi(k) |k\rangle$$

$$\downarrow$$

$$|\psi\rangle = \hat{1} \cdot |\psi\rangle = \int_{-\infty}^{+\infty} dx \, \varphi(x) |x\rangle$$

Postulate I

The state of an isolated physical system is represented, at a fixed time t, by a state vector $|\psi\rangle$ belonging to a Hilbert space $\mathcal{H}$ called the *state space*.

Postulate II.a

Every measurable physical quantity $\mathcal{A}$ is described by a Hermitian operator A acting in the state space $\mathcal{H}$. This operator is an observable, meaning that its eigenvectors form a basis for $\mathcal{H}$. The result of measuring a physical quantity $\mathcal{A}$ must be one of the eigenvalues of the corresponding observable A.

$$\left\{ \begin{array}{l} q_l \longrightarrow \hat{X}_l \\ p_l \longrightarrow \hat{P}_l \end{array} \right. \Rightarrow \begin{array}{l} [\hat{X}_l , \hat{P}_k] = i\hbar \, \delta_{lk} \\ F(q_l, p_l) = \hat{F}(\hat{X}_l, \hat{P}_l) \end{array}$$

Postulate II.b

When the physical quantity $\mathcal{A}$ is measured on a system in a normalized state $|\psi\rangle$, the probability of obtaining an eigenvalue (denoted a_n for discrete spectra and α for continuous spectra) of the corresponding observable A is given by the *amplitude squared* of the appropriate wave function (projection onto corresponding eigenvector).

$$\mathbb{P}(a_n) = |\langle a_n|\psi\rangle|^2 \qquad \text{(Discrete, nondegenerate spectrum)}$$

$$\mathbb{P}(a_n) = \sum_i^{g_n} |\langle a_n^i|\psi\rangle|^2 \qquad \text{(Discrete, degenerate spectrum)}$$

$$d\mathbb{P}(\alpha) = |\langle \alpha|\psi\rangle|^2 d\alpha \qquad \text{(Continuous, nondegenerate spectrum)}$$

Postulate II.c

If the measurement of the physical quantity $\mathcal{A}$ on the system in the state $|\psi\rangle$ gives the result a_n, then the state of the system immediately after the measurement is the normalized projection of $|\psi\rangle$ onto the eigensubspace associated with a_n

$$\psi \quad \stackrel{a_n}{\Longrightarrow} \quad \frac{P_n|\psi\rangle}{\sqrt{\langle\psi|P_n|\psi\rangle}}$$

Postulate III

The time evolution of the state vector $|\psi(t)\rangle$ is governed by the Schrödinger equation, where $H(t)$ is the observable associated with the total energy of the system (called the Hamiltonian)

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle$$

完结撒花