Green's Function

1. What is Green's Function:

As I have shown before, if we take Green's function as generalized function related to linear operator L, we have

$$L(G) = \delta$$

And if $L(y) = f$, we can obtain that

$$\langle y| = f * \delta$$

Now for convenience, we define that

$$LG(x,t) = \delta(x-t)$$

And there's $y = \int G(x,t) f(t)\,dt$

Since $Ly(x) = \int LG(x,t)\, f(t)\,dt = \int \delta(x-t) f(t)\,dt = f(x)$

Notice that we haven't taken boundary condition into account. If there's related items in y (like $\iint_{\Sigma} \varphi(\vec{r}_0) \frac{\partial G(\vec{r}, \vec{r}_0)}{\partial n_0} dA_0$), $Ly(x)$ won't be influenced at all, while boundary condition could be met now.

2. One dimension problems.

2.1 Some properties

Now we will think about the problem:

$$Ly = \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)\cdot y = f(x)$$

Here we have to deal with the equation:

$$\frac{d}{dx}\left[p(x)\frac{dG(x,t)}{dx}\right]+q(x)G(x,t)=\delta(x-t)$$

And we integrate it from $(x-\varepsilon)$ to $(x+\varepsilon)$:

$$p(x)\cdot\frac{dG(x,t)}{dx}\Big|_{t-\varepsilon}^{t+\varepsilon}+\int_{t-\varepsilon}^{t+\varepsilon}q(x)G(x,t)\,dx=1$$

We can't accept that $G(x,t)$ isn't continuous, but ~~if~~ it's acceptable if $\frac{dG(x,t)}{dx}$ is discontinuous. Thus,

$$\lim_{\varepsilon\to 0^+}\left[\frac{dG(x,t)}{dx}\Big|_{x=t+\varepsilon}-\frac{dG(x,t)}{dx}\Big|_{x=t-\varepsilon}\right]=\frac{1}{p(t)}$$

Then we will talk about the expansions of $G(x,t)$ in the eigenfunctions of operator L. Of course, L have to be Hermitian and consequently its eigenfunctions could be chosen orthonormal.

That is: $L\varphi_n(x)=\lambda_n\varphi_n(x)$, $\langle\varphi_n|\varphi_m\rangle=\delta_{mn}$

Notice that $\varphi_n(x)$ is complete, which means

$$|f\rangle=\sum\frac{\langle\varphi_n|f\rangle}{\langle\varphi_n|\varphi_n\rangle}|\varphi_n\rangle$$

That's $\delta(x-t)=\sum_m\varphi_m(x)\varphi_m^*(t)$

(Proof: $\langle x'|x\rangle=\sum\frac{\langle\varphi_n|x\rangle}{\langle\varphi_n|\varphi_n\rangle}\langle x'|\varphi_n\rangle=\sum\varphi_n(x')\varphi_n^*(x)$)

Assume that $G(x,t)=\sum_{mn}g_{nm}\varphi_n(x)\varphi_m^*(t)$

$LG = \delta$

$\therefore L \sum g_{nm} \varphi_n(x) \varphi_m^*(t) = \sum_m \varphi_m(x) \varphi_m^*(t)$

$\therefore \sum \lambda_n g_{nm} \varphi_n(x) \varphi_m^*(t) = \sum \varphi_m(x) \varphi_m^*(t)$

$\therefore G(x,t) = \sum_n \frac{\varphi_n^*(t)\varphi_n(x)}{\lambda_n}$

Thus, it's obvious that $G(x,t) = G(t,x)^*$

Example: $\begin{cases} Ly = \frac{d^2y}{dx^2} + y = f(x) \\ y(0) = 0 \\ y'(0) = 0 \end{cases} \Rightarrow \begin{cases} \frac{d^2G(x,t)}{dx^2} + G(x,t) = \delta(x-t) \\ G(0,t) = 0 \\ G'(0,t) = 0 \end{cases}$

Notice that Green's function for $x<t$ is $G(x,t) = 0$

As $y(0) = 0$, $y'(0) = 0 \Rightarrow G(0,t) = 0$; $G'(0,t) = 0$

But when $x>t$, it's not 0.

Assume that it's $G(x,t) = C_1(t)\sin x + C_2(t)\cos x \quad x>0$

and $\begin{cases} G(t_-,t) = G(t_+,t) \\ \frac{\partial G}{\partial x}(t_+,t) - \frac{\partial G}{\partial x}(t_-,t) = \frac{1}{p(t)} = 1 \end{cases}$

$\therefore G(x,t) = \cos t \sin x - \sin t \cos x = \sin(x-t) \quad , x>t$

Thus, $y(x) = \int_0^\infty G(x,t) f(t)\, dt = \int_0^x \sin(x-t) f(t)\, dt$

3. Green's function without boundary conditions or with boundary conditions

3.1 Wave Equation

Now we will talk about the equations:

$$\begin{cases} \frac{\partial^2 w}{\partial t^2} = a^2\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right) \\ w|_{t=0} = \phi(x, y, z) \\ \frac{\partial w}{\partial t}\Big|_{t=0} = \psi(x, y, z) \end{cases}$$

Notice that it's different from previous content. The previous are mainly about the influence of source. But note that it's wave equation now, so the transmission of the wave also ~~cou~~ counts. Note that due to superposition principle, we can devide it into two set of equations

Assume that $w(\vec{r}, t) = u(\vec{r}, t) + v(\vec{r}, t)$

and
$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) \\ u|_{t=0} = 0 \\ \frac{\partial u}{\partial t}\Big|_{t=0} = \psi(x, y, z) \end{cases} \quad ①$$

$$\begin{cases} \frac{\partial^2 v}{\partial t^2} = a^2\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) \\ v|_{t=0} = \phi(x, y, z) \\ \frac{\partial v}{\partial t}\Big|_{t=0} = 0 \end{cases} \quad ②$$

For ①, we should have the solution

$$u(\vec{r}, t) = \iiint_{\infty} \psi(\vec{r_0})\, G(\vec{r}, \vec{r_0}, t)\, d\vec{r_0}$$

And G is aimed to the influence of $\frac{\partial u}{\partial t}$. So we have

$$\begin{cases} \frac{\partial^2 G}{\partial t^2} = a^2\left(\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} + \frac{\partial^2 G}{\partial z^2}\right) \\ G|_{t=0} = 0 \\ \frac{\partial G}{\partial t}\Big|_{t=0} = \delta(x-x_0)\delta(y-y_0)\delta(z-z_0) \end{cases}$$

After Fourier transformation, we have

$$G(r,t) = \frac{1}{4\pi a r}[\delta(r-at) - \delta(r+at)] \quad , r = |\vec{r} - \vec{r_0}|$$

notice that $t > 0$

$$\therefore G(r,t) = \frac{1}{4\pi a} \cdot \frac{\delta(|\vec{r}-\vec{r_0}| - at)}{|\vec{r}-\vec{r_0}|}$$

$$\therefore u(\vec{r},t) = \frac{1}{4\pi a}\iiint_{\infty} \psi(\vec{r_0}) \frac{\delta(|\vec{r}-\vec{r_0}| - at)}{|\vec{r}-\vec{r_0}|} d\vec{r_0}$$

As for v, we have

$$\begin{cases} \frac{\partial^2 v}{\partial t^2} = a^2\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) & (a) \\ v|_{t=0} = \phi(x,y,z) & (b) \\ \frac{\partial v}{\partial t}\Big|_{t=0} = 0 & (c) \end{cases}$$

Now we set $\begin{cases} \frac{\partial^2 h}{\partial t^2} = a^2\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2}\right) \\ h|_{t=0} = 0 \\ \frac{\partial h}{\partial t}\Big|_{t=0} = \phi(x,y,z) \end{cases}$

we have $v = \frac{\partial h}{\partial t}$

Obviously (a) and (b) is satisfied by $\frac{\partial h}{\partial t}$.

Then $\frac{\partial v}{\partial t} = \frac{\partial^2 h}{\partial t^2} = a^2\nabla^2 h(x,y,z,t)|_{t=0} = a^2\nabla^2 h(x,y,z,0)$

$= a^2\nabla^2 0 = 0$

$\therefore v = \frac{\partial h}{\partial t} = \frac{\partial}{\partial t}\iiint \phi(\vec{r_0})\, G(\vec{r}, \vec{r_0}, t)\, d\vec{r_0}$

$$\therefore u(\vec{r}, t) = \frac{1}{4\pi a}\left[\frac{\partial}{\partial t}\iiint \phi(\vec{r_0})\frac{\delta(|\vec{r}-\vec{r_0}|-at)}{|\vec{r}-\vec{r_0}|}\, d\vec{r_0} + \frac{1}{4\pi a}\iiint \psi(\vec{r_0})\frac{\delta(|\vec{r}-\vec{r_0}|-at)}{|\vec{r}-\vec{r_0}|}\, d\vec{r_0}\right]$$

3.2 Pission equation:

We know that $\iint (u\nabla v - v\nabla u)\cdot ds = \iiint (u\nabla^2 v - v\nabla^2 u)dv$

As before, we have $\nabla^2 v(\vec{r}, \vec{r_0}) = \delta(\vec{r}-\vec{r_0})$

and $\nabla^2 u = f(\vec{r})$

$$\therefore u(\vec{r_0}) = \iiint_T v(\vec{r}, \vec{r_0}) f(\vec{r})\, dv - \iint_\Sigma \left[v(\vec{r}, \vec{r_0})\frac{\partial u(\vec{r})}{\partial n} - u(\vec{r})\frac{\partial v(\vec{r}, \vec{r_0})}{\partial n}\right] ds$$

Then we will talk about boundary conditions

(i) For Dirchilet condition, we know the distribution of u on Σ, and if we require that $v|_\Sigma = 0$, then the item related to $\frac{\partial u}{\partial n}$ would be 0.

and now we have:

$$u(\vec{r}_0) = \iiint G(\vec{r},\vec{r}_0)f(\vec{r})\,dv + \iint_{\Sigma} \varphi(\vec{r})\frac{\partial G(\vec{r},\vec{r}_0)}{\partial n}\,ds$$

If the boundary condition is

$$\left[\alpha\frac{\partial u}{\partial n}+\beta u\right]_{\Sigma}=\varphi$$

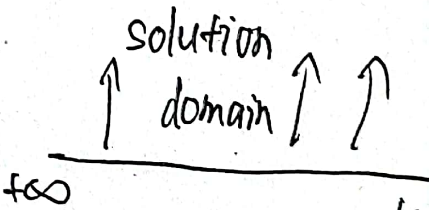
we will give v the same: $\left[\alpha\frac{\partial v}{\partial n}+\beta v\right]_{\Sigma}=0$

Thus, $$v\frac{\partial u}{\partial n}-u\frac{\partial v}{\partial n} = \frac{1}{\alpha}\left[\left(\alpha v\frac{\partial u}{\partial n}+\beta uv\right)-\left(\alpha u\frac{\partial v}{\partial n}+\beta uv\right)\right] = \frac{1}{\alpha}v\varphi$$

$$\therefore u(\vec{r}_0) = \iiint_T v(\vec{r},\vec{r}_0)f(\vec{r})\,dv - \frac{1}{\alpha}\iint_{\Sigma} v(\vec{r},\vec{r}_0)\varphi(\vec{r})\,ds$$

As for Neumann condition, we hope that

$$\begin{cases}\nabla^2 v = \delta(\vec{r}-\vec{r}_0) \quad (*)\\ \left.\frac{\partial G}{\partial n}\right|_{\Sigma}=0\end{cases}$$, and thus we have



$$u(\vec{r}_0) = \iiint_T v(\vec{r},\vec{r}_0)f(\vec{r})\,dr - \iint_{\Sigma} v(\vec{r},\vec{r}_0)\varphi(\vec{r})\,ds \qquad \left(\frac{\partial u(\vec{r})}{\partial n}=\varphi(\vec{r})\right)$$

If the solution domain is semi-infinite, it's ok. But what if it's closed? You will find that (*) has no solution!

$\nabla^2 v = \delta(\vec{r}-\vec{r}_0)$ and $\left.\frac{\partial G}{\partial n}\right|_{\Sigma}=0$ is contradictory

Now we have to introduce

$$\nabla^2 G = \delta(x-x_0)\,\delta(y-y_0)\,\delta(z-z_0) - \frac{1}{V_T} \to \text{volume/area} \cdots$$

and $\left.\frac{\partial G}{\partial n}\right|_\Sigma = 0$

and we have

$$u(\vec{r}_0) = \iiint G(\vec{r}, \vec{r}_0) f(\vec{r})\, d\tau - \iint_\Sigma G(\vec{r}, \vec{r}_0)\, \varphi(\vec{r})\, ds + \frac{\iiint u(\vec{r})\, d\vec{r}}{V_T}$$

that's the same as adding a constant. And we can find it out easily

Now we will talk about boundary conditions. Now we have to generalize Green's function. For the equation.

$$a_{11} u_{xx} + 2a_{12} u_{xy} + a_{22} u_{yy} + b_1 u_x + b_2 u_y + cu = 0$$

We denote it as $Lu = 0$

If there's M such that

$$vLu - uMv = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}, \quad \text{and } X = X(x,y),\ Y = Y(x,y)$$

we could call M as adjoint operator of L, and vice versa. If $M = L$, then it's self-adjoint operator.

Now we have

$$\iint_T (vLu - uMv)\, ds = \int_\Sigma [X \cos\langle \vec{n}, \vec{x} \rangle + Y \cos\langle \vec{n}, \vec{y} \rangle]\, ds$$

This generalized Green's function.

Example: $U_{tt} - a^2\nabla^2 U = -f(\vec{r})e^{i\omega t}$

we assume that ~~tt~~ $U(\vec{r},t) = u(\vec{r})e^{i\omega t}$,

and there's $(\omega^2 + a^2\nabla^2)u = -f(\vec{r})$

The green's function is, in fact

$$G(\vec{r},\vec{r_0}) = \frac{1}{4\pi} e^{i\frac{\omega}{a}|\vec{r}-\vec{r_0}|}\frac{1}{|\vec{r}-\vec{r_0}|}$$

and we have

$$u(\vec{r}) = \frac{1}{4\pi}\iint_{\Sigma}\left[\frac{e^{i\frac{\omega}{a}|\vec{r}-\vec{r_0}|}}{|\vec{r}-\vec{r_0}|}\frac{\partial u}{\partial n} - u\frac{\partial}{\partial n}\frac{e^{i\frac{\omega}{a}|\vec{r}-\vec{r_0}|}}{|\vec{r}-\vec{r_0}|}\right]ds_0$$

$$-\frac{1}{4\pi}\iiint_{T}\frac{e^{i\frac{\omega}{a}|\vec{r}-\vec{r_0}|}}{|\vec{r}-\vec{r_0}|}f(\vec{r}_0)\,dv_0$$

3.3 Green Function with time

Now we have
$$\begin{cases} u_{tt} - a^2\nabla^2 u = f(\vec{r},t) \\ \left(\alpha\frac{\partial u}{\partial n} + \beta u\right)\Big|_{\Sigma} = \theta(M,t) \\ u|_{t=0} = \varphi(\vec{r}),\ u_t|_{t=0} = \psi(\vec{r}) \end{cases}$$

We assume that
$$\begin{cases} G_{tt} - a^2\nabla^2 G = \delta(\vec{r}-\vec{r_0})\,\delta(t-t_0) \\ \alpha\frac{\partial G}{\partial n} + \beta G\,\Big|_{\Sigma} = 0 \\ G|_{t=0} = 0,\ G_t|_{t=0} = 0 \end{cases}$$

Notice that the symmetry is:

$$G(\vec{r}, t; \vec{r_0}, t_0) = G(\vec{r_0}, -t_0; \vec{r}, -t)$$

So we have to:
$$\begin{cases} u_{t_0t_0}(\vec{r_0}, t_0) - a^2\nabla_0^2 u(\vec{r_0}, t_0) = f(\vec{r_0}, t_0) \\ \alpha\frac{\partial u_0}{\partial n_0} + \beta u_0 = \theta(M_0, t_0) \\ u(\vec{r_0}, t_0)\big|_{t_0=0} = \varphi(\vec{r_0}), \quad u_{t_0}(\vec{r_0}, t_0)\big|_{t_0=0} = \psi(\vec{r_0}) \end{cases}$$

and
$$\begin{cases} G_{t_0t_0}(\vec{r}, t; \vec{r_0}, t_0) - a^2\nabla_0^2 G(r, t; r_0, t_0) = \delta(r-r_0)\delta(t-t_0) \\ \left[\alpha\frac{\partial G(r, t; r_0, t_0)}{\partial n_0} + \beta G(r, t; r_0, t_0)\right]\Big|_\Sigma = 0 \\ G(\vec{r}, t; \vec{r_0}, t_0)\big|_{t_0=0} = 0, \quad G_{t_0}(\vec{r}, t; \vec{r_0}, t_0)\big|_{t_0=0} = 0 \end{cases}$$

$$\therefore \iiint_T \int_0^{t-\varepsilon} (G u_{t_0t_0} - u G_{t_0t_0}) dv_0 dt_0 - a^2 \iiint_T \int_0^{t+\varepsilon} (G\Delta_0 u - u\Delta_0 G) dv_0 dt_0$$

$$= \iiint_T \int_0^{t+\varepsilon} G(\vec{r}, t; \vec{r_0}, t_0) f(\vec{r_0}, t_0) dv_0 dt_0 - \iiint_T \int_0^{t+\varepsilon} u\,\delta(\vec{r}-\vec{r_0})\delta(t-t_0) dv_0 dt_0$$

$(\varepsilon \to 0)$ ε: $\int \delta(t-t_0) dt_0$ is certain

$$\therefore u(\vec{r}, t) = \iiint_T \int_0^{t+\varepsilon} G(\vec{r}, t; \vec{r_0}, t_0) f(\vec{r_0}, t_0) dv_0 dt_0 - \iiint_T \int_0^{t+\varepsilon} (G u_{t_0t_0} - u G_{t_0t_0}) dv_0 dt_0$$
$$+ a^2 \iiint_T \int_0^{t+\varepsilon} (G\Delta_0 u - u\Delta_0 G) dv_0 dt_0$$

Notice that $G u_{t_0t_0} - u G_{t_0t_0} = \frac{d}{dt_0}(G u_{t_0} - u G_{t_0})$

and when $t < t_0$, $G = 0$, $G_{t_0} = 0$

So we can obtain:

$$u(\vec{r}, t) = \iiint_T \int_0^t G(\vec{r}, t, \vec{r_0}, t_0) f(\vec{r_0}, t_0) dv_0 dt_0 + a^2 \iint_\Sigma \int_0^t \left(G\frac{\partial u}{\partial n_0} - u\frac{\partial G}{\partial n_0}\right) ds_0 dt_0$$
$$- \iiint_T [G u_{t_0} - u G_{t_0}]\big|_{t_0=0} dv_0$$

As for the problem

$$\begin{cases} u_t - a^2 \Delta u = f(\vec{r}, t) \\ \left[\alpha \frac{\partial u}{\partial n} + \beta u\right]\Big|_{\Sigma} = \Theta(M, t) \\ u|_{t=0} = \varphi(\vec{r}) \end{cases}$$

We can obtain that

$$u(\vec{r}, t) = \iiint_{T} \int_0^t G(\vec{r}, t; \vec{r}_0, t_0) f(\vec{r}_0, t_0)\, dv_0\, dt_0$$
$$+ a^2 \iint_{\Sigma} \int_0^t \left(G \frac{\partial u}{\partial n_0} - u \frac{\partial G}{\partial n_0}\right) dS_0\, dv_0$$
$$+ \iiint_{T} [uG]\big|_{t_0=0}\, dV_0$$

4. How to find out the function

4.1 One dimension $(a \le x \le b)$

Now we will talk ~~out~~ about the question where L is the form of $\frac{d}{dx}[P(x)\frac{d}{dx}] + q(x)$ with homogeneous boundary condition. As is shown before, we have

$$\begin{cases} G(t_-, t) = G(t_+, t) \\ \frac{\partial G}{\partial x}(t_+, t) - \frac{\partial G}{\partial x}(t_-, t) = \frac{1}{p(t)} = 1 \end{cases}$$

Given $y(x) = \int_a^b G(x, t) f(t)\, dt$, if boundary condition is $Ly = 0$ (L is linear), it's obvious that G also ~~st~~ satisfy the condition.

ow we set that $y_1(x)$ is the solution to $L = 0$ and also atisfies the boundary condition ~~of~~. at $x=a$; while $y_2(x)$ satisfies the boundary condition at $x=b$

So the most general $G(x,t)$ should be

$$G(x,t) = \begin{cases} y_1(x)h_1(t), & x<t \\ y_2(x)h_2(t), & x>t \end{cases}$$

It satisfies boundary condition and the former equations as ell. So we have $G(x,t) = \begin{cases} Ay_1(x)y_2(t), & x<t \\ Ay_2(x)y_1(t), & x>t \end{cases}$

Thus, $A[y_2'(t)y_1(t) - y_1'(t)y_2(t)] = \frac{1}{P(t)}$

and $A = \dfrac{1}{P(t)\cdot[y_2'(t)y_1(t) - y_1'(t)y_2(t)]}$

Notice that $\begin{cases} P(t)y_1''(t) + P'(t)y_1'(t) + q(t)y_1(t) = 0 \\ P(t)y_2''(t) + P'(t)y_2'(t) + q(t)y_2(t) = 0 \end{cases}$

$\therefore P'(t)\cdot(y_2'(t)y_1(t) - y_1'(t)y_2(t)) + P(t)\frac{d}{dt}(y_2'(t)y_1(t) - y_1'(t)y_2(t)) = 0$

$\therefore y_2'(t)y_1(t) - y_1'(t)y_2(t) = \frac{C}{P(t)}$, and $A = C$ (there's not t)

and we have

$$y(x) = Ay_2(x)\int_a^x y_1(t)f(t)dt + Ay_1(x)\int_0^x y_2(t)f(t)dt$$

(*): verify that it is the solution

2 More dimensions

4.2.1 Electric image method

Now we will focus on Possion's equation with Dirichlet Conditions

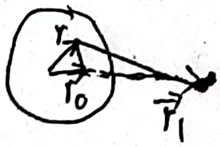
That is to say, $\begin{cases} \Delta G = \delta(\vec{r} - \vec{r}_0) \\ G|_{\Sigma} = 0 \end{cases}$

And when it comes to electrostatic problems, we have

$$\begin{cases} \Delta \varphi = -\frac{e}{\varepsilon_0}\delta(\vec{r} - \vec{r}_0) \text{ (point charge)} \\ \varphi|_{\Sigma} = 0 \end{cases}$$

They are literally the same! So we can apply the method to Green's function. $(\varepsilon_0 = 1,\ e = -1)$

Example: inside a ball $r=a$: $\begin{cases} \Delta^3 u = 0 \quad (r<a) \\ u|_{r=a} = f(\theta, \varphi) \end{cases}$



Solution: taking use of electric image method, and take $\begin{cases} \varepsilon_0 = 1 \\ e = -1 \end{cases}$,

we have $G(\vec{r}, \vec{r}_0) = -\frac{1}{4\pi}\frac{1}{|\vec{r} - \vec{r}_0|} + \frac{a}{r_0}\frac{1}{4\pi}\cdot\frac{1}{|\vec{r} - \vec{r}_1|}$, $\vec{r}_1 = \frac{a^2}{r_0^2}\vec{r}_0$

Now we have $u(\vec{r}) = \iiint\limits_{T} G(\vec{r}, \vec{r}_0) f(\vec{r}_0) dV_0 + \iint\limits_{\Sigma} \varphi(\vec{r}_0) \frac{\partial G(\vec{r}, \vec{r}_0)}{\partial n_0} dS_0$

And also we have

$$\frac{\partial G}{\partial n_0}\Big|_\Sigma = \frac{a^2-r^2}{4\pi a|\vec{r}-\vec{r_0}|^3}$$

$$\therefore u = \frac{a}{4\pi}\int_0^{\pi}\int_0^{2\pi} f(\theta_0,\varphi_0)\,\frac{a^2-r^2}{(a^2-2ar_0\cos\Theta+r_0^2)^{\frac{3}{2}}}\sin\theta_0\,d\theta_0\,d\varphi_0$$

$$\left(\text{or } \frac{1}{4\pi a}\int_0^{\pi}\int_0^{2\pi} f(\theta_0,\varphi_0)\cdot\frac{(a^2-r^2)a^2}{|\vec{r}-\vec{r_0}|^3}\sin\theta_0\,d\theta_0\,d\varphi_0\right)$$

4.2.2 Impulse Theorem:

Now we will talk about the Green function in the form of wave ~~func~~ equation or heat conduction equation.

For example:
$$\begin{cases} G_{tt}-a^2G_{xx}=\delta(x-\xi)\delta(t-\tau) \\ G|_{t=0}=0 \quad G_t|_{t=0}=0 \end{cases}$$

It could be transformed into

$$\begin{cases} G_{tt}-a^2G_{xx}=0 \\ G|_{t=0}=0 \\ G_t|_{t=0}=\delta(x-\xi) \end{cases}$$, and we have

$$G(x,t;\xi,\tau) = \frac{1}{2a}\int_{x-a(t-\tau)}^{x+a(t-\tau)}\delta(\xi_0-\xi)\,d\xi_0 = \begin{cases} 0, & \xi<x-a(t-\tau) \text{ or } x+a(t-\tau)<\xi \\ \frac{1}{2a}, & x-a(t-\tau)<\xi<x+a(t-\tau) \end{cases}$$

$$\text{and } u(x,t) = \int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)}\frac{1}{2a}f(\xi,\tau)\,d\xi\,d\tau$$

4.2.3 Other ways

Example: $\nabla^2 G(\vec{r}_1, \vec{r}_2) + k^2 G(\vec{r}_1, \vec{r}_2) = \delta(\vec{r}_1 - \vec{r}_2)$

Solution: we set $G(\vec{r}_1, \vec{r}_2) = y(r)$, $r = |\vec{r}_1 - \vec{r}_2|$

$$\therefore \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial y}{\partial r}\right) + k^2 y = \delta(r)$$

Now we assume that $y = \frac{f}{r}$, and we get

$$\frac{\frac{d^2 f}{dr^2} + k^2 f}{r} = \delta(r)$$

$$\therefore \frac{d^2 f}{dr^2} + k^2 f = 0, \quad \text{and } f = A\exp(ikr)$$

Given $\iiint_{|r|<\varepsilon} y(r)\, dv = -\frac{A}{r^2} \cdot 4\pi r = 1$ $\therefore A = -\frac{1}{4\pi}$,

and $y = -\frac{\exp[ik|\vec{r}_1 - \vec{r}_2|]}{4\pi|\vec{r}_1 - \vec{r}_2|}$, that's Green's Function.