代数基础选讲.

1. Determinants.

1.1 ~~齐次~~ / Homogeneous Linear equations

Crammer's rule

1.2. Levi-Civita

$$\varepsilon_{ij\cdots} = \begin{cases} +1 & ij\cdots \text{ an even permutation} \\ -1 & ij\cdots \text{ an odd permutation} \\ 0 & \end{cases}$$

$$D_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ - & - & & - \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{ij\cdots} \varepsilon_{ij\cdots} a_{1i} a_{2j} \cdots$$

$$\cancel{\varepsilon_{ij\cdots} = \begin{vmatrix} \delta_{11} & \delta_{12} & \cdots & \delta_{1n} \end{vmatrix}}$$

$$\varepsilon_{i_1 i_2 \cdots i_n} = \varepsilon_{j_1 \cdots j_n} \delta_{i_1 j_1} \delta_{i_2 j_2} \cdots \delta_{i_n j_n}$$

$$= \begin{vmatrix} \delta_{i_1 1} & \delta_{i_2 1} & \cdots & \delta_{i_n 1} \\ - & - & & - \\ \delta_{i_1 n} & \delta_{i_2 n} & \cdots & \delta_{i_n n} \end{vmatrix}.$$

1.3 Properties

1.4. Minors

$$D_n = \sum_{j=1}^{n} a_{ij} \underbrace{(-1)^{i+j} M_{ij}}_{\text{cofactor}}$$

Minors

1.5. Equations.

2. Matrix / Matrices

2.1. $A+B$, $\underline{AB}$, αA

2.2. $AB \neq BA$, For squares, commutator 对易子

$$[A, B] = AB - BA$$

(~~Poisson~~ Poisson Bracket)

$$\{u, v\} = \sum_i \left(\frac{\partial u}{\partial q_i} \frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i} \frac{\partial v}{\partial q_i} \right)$$

1. $\{u, v\} = -\{v, u\}$
2. $\{u, c\} = 0$
3. $\{u_1 + u_2, v\} = \{u_1, v\} + \{u_2, v\}$
4. $\{u, v_1 + v_2\} = \{u, v_1\} + \{u, v_2\}$
5. $\{u_1 u_2, v\} = \{u_1, v\} u_2 + u_1 \{u_2, v\}$
6. $\{u, v_1 v_2\} = \{u, v_1\} v_2 + v_1 \{u, v_2\}$
7. $\{u, \{v, w\}\} + \{v, \{w, u\}\} + \{w, \{u, v\}\} = 0.$

$$[u, v] = D\{u, v\}$$

例 $[\hat{x}, \hat{p}_x] = \hat{x}\hat{p}_x - \hat{p}_x\hat{x} = D\{x, p_x\}$

$$= D\left(\frac{\partial x}{\partial x} \frac{\partial p_x}{\partial p_x} + \cdots \right) = D = i\hbar.$$

$(AB)C = A(BC)$ ✓

Example 1. Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_1 \sigma_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \sigma_2 \sigma_1 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$\Rightarrow [\sigma_1, \sigma_2] = 2i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2i\sigma_3.$$

Example 2.

$$A = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad B = (4 \ 5 \ 6)$$

$$AB = \begin{pmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{pmatrix} \quad BA = (4\times1 + 2\times5 + 3\times6) = 32 = tr(AB)$$

$$(AB)^n = A\underbrace{BABA\cdots}B = tr(AB)^{n-1} AB.$$

3. Unit Matrix, Diagonal Matrices

3.1 Matrix Inverse, $|A| \neq 0$, ~~~~ $\exists! B \in R^{n\times n}$

$$AB = BA = \mathbb{1}, \quad B =: A^{-1}$$

Example $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$, $AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Matrix is a ring. ⁞ $\nexists$ nonzero commutative ring in which every nonzero element has a multiplicate inverse

is a field

$\cancel{(A^{-1})_{ij}}$ $(A^{-1})_{ij} = \dfrac{(-1)^{i+j} M_{ji}}{\det(A)}$

$\left((-1)^{i+j} M_{ji}\right) = A^*$ ~~Conjugated~~ $\cancel{A_a}$ Adjugate matrix.

Example Gauss – Jordon matrix inversion

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix} \to \left(\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & & & \\ & 1 & & \\ & & 1 & \bigcirc \end{array}\right)$$

3.2. Derivative of Determinants

$$\frac{\partial \det(A)}{\partial a_{ij}} = (-1)^{i+j} M_{ij} = (A^{-1})_{ji} \det(A)$$

$$\frac{d(\det A)}{dx} = \det(A) \sum_{ij} (A^{-1})_{ji} \frac{d a_{ij}}{dx}.$$

3.3 $\det(AB) = \det(A)\det(B)$.

3.4. rank of Matrix.

3.5.

3.5.1 Transpose. $(\tilde{A})_{ij} = a_{ji}$

$\tilde{A} = A$ symmetric, anti ---

complex conjugate, $\bar{A} = (\overline{a_{ij}})$

3.5.2. adjoint of matrix (伴随，共轭转置).

$(A^\dagger)_{ij} = \overline{a_{ji}}$

3.5.3. trace

~~tarc~~ $tr(A) = \sum_{i=1}^{n} a_{ii}$

trace is l 0

determinant is not l 0.

$tr(A+B) = tr(A) + tr(B)$

$tr(AB) = (AB)_{ii} = a_{ij} b_{ji} = b_{ji} a_{ij} = (BA)_{jj} = tr(BA)$

$tr([A,B]) = 0$

$tr(ABC) = tr(BCA) = tr(CAB) \neq \ldots$

3.5.4 Operations on matrix product.

$(AB)^T = B^T A^T$

$(AB)^\dagger = B^\dagger A^\dagger$

$(AB)^{-1} = B^{-1} A^{-1}$

3.5.5. Orthogonal Matrices.

$S^{-1} = S^T$ or $S^T S = \mathbb{1}$ $\quad \begin{pmatrix} | & | & | \end{pmatrix}$

$\det(S) = \pm 1$

$\det(S) = 1$, 转动 proper

$\det(S) = -1$, 镜面，转动 improper

$S_1 S_2$, orthogonal

3.5.6. Unitary Matrices (酉、幺正矩阵)

$U^\dagger = U^{-1}$, $U^\dagger U = U U^\dagger = \mathbb{1}$.

$\det(U)\det(U^\dagger) = |\det(U)|^2 = 1$

$\det(U) = e^{i\theta}$, $\det(U^\dagger) = e^{-i\theta}$, $U_1 U_2$ Unitary

3.5.7. Hermitian Matrice (Self-Adjoint)

$(H^\dagger)_{ij} = (H)_{ij} \to \bar{h}_{ji} = h_{ij}$ $\quad \begin{pmatrix} \ddots & & \\ & \ddots & \\ & & \ddots \end{pmatrix}$

Quantum Mechanic, $A \to \hat{A}$

A, B, Hermitian.

AB not necessary,

$AB + BA$ ✓

$AB - BA \to$ anti-Hermitian

$(AB - BA)^\dagger = -(AB - BA)$

3.5.8. Direct Product

$A_{m\times n}$, $B_{m'\times n'}$

$C = A\otimes B$, $C_{mm'\times nn'}$

$C_{\alpha\beta} = A_{ij}B_{kl}$

$$\begin{cases} \alpha = m'(i-1)+k \\ \beta = n'(j-1)+l \end{cases}$$

Example 3.

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}$

$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix}$ $\Rightarrow$ $A\otimes B = \begin{pmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & | \\ a_{11}b_{21} & a_{11}b_{22} & | \\ & & | \end{pmatrix}$

$C = A\otimes B$, $C' = A'\otimes B'$

$CC' = (AA')\otimes(BB')$

$C\otimes(A+B) = C\otimes A + C\otimes B$, $(A+B)\otimes C = A\otimes C + B\otimes C$,

Example 4 two component wave function

$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}$. $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}$ $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}$

$\sigma_i^2 = \mathbb{1}_2$, $\sigma_i\sigma_j + \sigma_j\sigma_i = 0$ $\forall i,j,\ i\neq j$

$E^2 = |\vec{p}|^2c^2 + m^2c^4$

$\sigma = \sigma_1\hat{e}_1 + \sigma_2\hat{e}_2 + \sigma_3\hat{e}_3$, $(\vec{\sigma}\cdot\vec{p}) = p^2\mathbb{1}_2$

$\Rightarrow$ $E^2\mathbb{1}_2 - c^2(\vec{\sigma}\cdot\vec{p})^2 = m^2c^4\mathbb{1}_2$

$(E\mathbb{1}_2 + c\sigma\cdot p)(E\mathbb{1}_2 - c\sigma\cdot p)\psi_1 = m^2c^4\psi$

Define
$$\begin{cases} (E\mathbb{1}_2 - c\sigma\cdot p)\psi_1 = mc^2\psi_2 \\ (E\mathbb{1}_2 + c\sigma\cdot p)\psi_2 = mc^2\psi_1 \end{cases}$$
[illegible].

则 $\psi_1 = \psi_A + \psi_B$, $\psi_2 = \psi_A - \psi_B$.

3.5.9 Functions of Matrices.

$$\exp(A) = \sum_{j=0}^{\infty}\frac{1}{j!}A^j$$

$$\sin(A) = \sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)!}A^{2j+1}$$

$$\cos(A) = \sum_{j=0}^{\infty}\frac{(-1)^j}{(2j)!}A^{2j}$$

For euler identity, Pauli

$\exp(i\sigma_k\theta) = \cos(\sigma_k\theta) + i\sin(\sigma_k\theta) = \mathbb{1}_2\cos\theta + i\sigma_k\sin\theta$.

For Dirac $\sigma^{\mu\nu}$

$\exp(i\sigma^{\mu\nu}\theta) = \mathbb{1}_4\cos\theta + i\sigma^{\mu\nu}\sin\theta$.

$\exp(i\sigma^{0k}\zeta) = \mathbb{1}_4\cosh\zeta + i\sigma^{0k}\sinh\zeta$.

Hermitian and Unitary matrices $\overset{\text{are}}{\text{is}}$ related

$U = \exp(iH) = e^{iH}$

U is Unitary , H is Hermitian.

$U^\dagger = e^{-iH^\dagger} = e^{-iH} = (e^{iH})^{-1} = U^{-1}$.

Trace formula for Hermitian Matrix.

$\det(\cancel{\exp}\, e^{H}) = e^{tr(H)}$. ([illegible]).

Example 5.

$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}$ $(\sigma_3)^n = \begin{pmatrix} 1 & 0 \\ 0 & (-1)^n\end{pmatrix}$

$e^{\sigma_3} = \begin{pmatrix} \sum_{n=0}^{\infty}\frac{1}{n!} & 0 \\ 0 & \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\end{pmatrix} = \begin{pmatrix} e & 0 \\ 0 & e^{-1}\end{pmatrix}$

Baker – Hausdorff formula. $e^{-T}Ae^{T} = A + [A,T] + \frac{1}{2!}[[A,T],T] + \frac{1}{3!}[[[A,T],T],T] + \cdots$

Proof: $f(\lambda) = e^{\lambda A} B e^{-\lambda A}$

$f(0) = B$

$f'(0) = \left(e^{\lambda A} A B e^{-\lambda A} + e^{\lambda A} B (-A) e^{-\lambda A}\right)_{\lambda=0} = \underline{e^{\lambda A}[A,B]e^{-\lambda A}}\Big|_{\lambda=0} = [A,B]$

$f''(0) = \left[e^{\lambda A} A [A,B] e^{-\lambda A} + e^{\lambda A}[A,B](-A)e^{-\lambda A}\right]_{\lambda=0} = [A,[A,B]]$

$f^{(k)}(0) = [A,\cdot]^k B.$

$f(\lambda) = \sum_{k=0}^{\infty} \frac{1}{k!}\lambda^k f^{(k)}(0) = \sum_{k=0}^{\infty}\frac{1}{k!}\lambda^k [A,\cdot]^k B$ ✓ $= e^{\lambda[A,\cdot]}B.$

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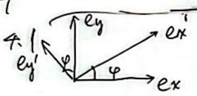
4. Coordinate Transformation → Rotation. 演示 $A\times(B\times C)$



~~e_x $e_x' = \cos\varphi\, e_x$~~

$e_x = \cos\varphi\, e_x' - \sin\varphi\, e_y'$

$e_y = \sin\varphi\, e_x' + \cos\varphi\, e_y'$ (1)

$\vec{A} = A_x e_x + A_y e_y$

$= (A_x\cos\varphi + A_y\sin\varphi)\, e_x' + (-A_x\sin\varphi + A_y\cos\varphi)\, e_y'$

$\vec{A}' = \begin{pmatrix} A_x' \\ A_y' \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} A_x \\ A_y \end{pmatrix}$

$\begin{pmatrix} A_x \\ A_y \end{pmatrix} = \begin{pmatrix} \cos(-\varphi) & \sin(-\varphi) \\ -\sin(-\varphi) & \cos(-\varphi) \end{pmatrix}\begin{pmatrix} A_x' \\ A_y' \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ +\sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} A_x' \\ A_y' \end{pmatrix}$

it $S = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$ $S' = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} = S^T$

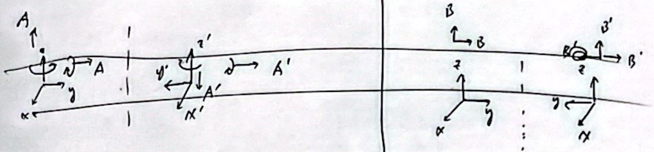
$\vec{A} = S'S\vec{A}$, $SS' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$S^{-1} = S^T$, orthogonal

By (1) $\vec{e}_x = (\vec{e}_x'\cdot\vec{e}_x)\,\vec{e}_x' + (\vec{e}_y'\cdot\vec{e}_x)\,\vec{e}_y'$

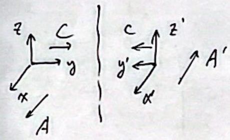
$\vec{e}_y = (\vec{e}_x'\cdot\vec{e}_y)\,\vec{e}_x' + (\vec{e}_y'\cdot\vec{e}_y)\,\vec{e}_y'$ $\Rightarrow$ $S = \begin{pmatrix} \vec{e}_x'\cdot\vec{e}_x & \vec{e}_x'\cdot\vec{e}_y \\ \vec{e}_x\cdot\vec{e}_y' & \vec{e}_y'\cdot\vec{e}_y \end{pmatrix}$

2. Reflection (improper)



~~$A' = \det(S)$~~ SA $B' = SB$ ✓

Vector $\begin{pmatrix} 1 & & \\ & -1 & \\ & & +1 \end{pmatrix}$



$C_x = A_y B_z - A_z B_y$

$\Rightarrow -C_x$

$C_y \rightarrow C_y$

$A' = \det S \cdot SA$

pseudo vector

$|V_3| = |V_1 + V_2|$

$|V_3'| = |V_1' - V_2'|$

$P + P = P$, $V + V = V$ $P + V$?

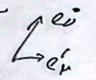
$P \times V = V$, $V \times V = P$, $P \times P = P$

$(RV_1)\times(RV_2) = (\det R)\,R\,(V_1 \times V_2)$

$A\cdot B\times C$ change sign, pseudoscalar

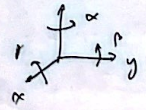
$A\times(B\times C)$ V.

3. Rotations in $\mathbb{R}^3$.



$e_\mu'\cdot e_\nu = \frac{\partial x_\nu}{\partial x_\mu'} = \frac{\partial x_\mu'}{\partial x_\nu}$

$S = \begin{pmatrix} e_1'\cdot e_1 & e_1'\cdot e_2 & e_1'\cdot e_3 \\ e_2'\cdot e_1 & e_2'\cdot e_2 & e_2'\cdot e_3 \\ e_3'\cdot e_1 & e_3'\cdot e_2 & e_3'\cdot e_3 \end{pmatrix}$



$S_1(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$S_2(\beta) = \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix}$$

$$S_3(\gamma) = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad S_1 S_2 \neq S_2 S_1$$

$$\Rightarrow S = S_3 S_2 S_1$$

$$= \begin{pmatrix} \cancel{\cos\gamma\cos\beta\cos\alpha - \sin\gamma\sin\alpha} & \cancel{\cos\gamma\cos\beta\sin\alpha} + \sin\gamma\cos\alpha \\ & \ddots \end{pmatrix}$$

4. Differential Vector Operators.

4.1 Gradient

$$d\varphi = \frac{\partial\varphi}{\partial x_1} dx_1 + \dots \quad \Rightarrow \nabla\varphi = \left(\frac{\partial\varphi}{\partial x_1}, \frac{\partial\varphi}{\partial x_2}, \frac{\partial\varphi}{\partial x_3}\right).$$

$$\vec{\nabla}\varphi \cdot d\vec{r} = d\varphi$$

$$S(\nabla\varphi) = \frac{\partial x_\mu}{\partial x_\nu'} \frac{\partial\varphi}{\partial x_\mu} = (\nabla\varphi)'$$

$$= \begin{pmatrix} \partial x_1/\partial x_1' & - & - \\ \vdots & \ddots & \\ \vdots & & \ddots \end{pmatrix} \begin{pmatrix} \partial\varphi/\partial x_1 \\ \partial\varphi/\partial x_2 \\ \partial\varphi/\partial x_3 \end{pmatrix}$$

Example.

$$\nabla r^n = n r^{n-1} \nabla r = n r^{n-1} \hat{r} \qquad \partial_i r^n \vec{e}_i$$

4.2. Divergence

$$\nabla A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \cdot \vec{r} = \partial_i r_i = 3$$

Example

$$\nabla \cdot f(r)\hat{r} = \frac{\partial}{\partial x}\left(\frac{x f(r)}{r}\right) + \dots$$

$$= \frac{f(r)}{r} - \frac{x f(r)}{r^2}\frac{\partial r}{\partial x} + \frac{x}{r}\frac{df(r)}{dr}\frac{\partial r}{\partial x} = f(r)\left[\frac{1}{r} - \frac{x^2}{r^3}\right] + \frac{x^2}{r^2}\frac{df(r)}{dr}$$

$$\nabla \cdot f(r)\hat{r} = 2\frac{f(r)}{r} + \frac{df(r)}{dr}$$

$$\partial_i f(r) \frac{r_i}{r} = \dots$$

$$\nabla \cdot r^n \hat{r} = (n+2) r^{n-1}$$

Alternatively,

$$\nabla \cdot f(r)\hat{r} = \frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial r}\left(r^2\sin\theta\, f(r)\right)\right) \cancel{+ \frac{\partial}{\partial\theta}} +$$

$$= \frac{1}{r^2}\left(2r f(r) + \frac{\partial f(r)}{\partial r} r^2\right) \checkmark$$

4.3. equation of continuity

$$\begin{cases} \dfrac{\partial\rho}{\partial t} + \nabla \cdot (\rho\vec{v}) = 0 \\ \dfrac{\partial w}{\partial t} + \nabla \cdot \vec{S} = 0 \\ \dfrac{\partial\rho}{\partial t} + \nabla \cdot \vec{J} = 0 \end{cases}$$

$\nabla \cdot B = 0 \Rightarrow$ solenoidal.

4.4 curl

$$\nabla \times V = \begin{vmatrix} e_x & e_y & e_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & V_z \end{vmatrix} \qquad \frac{\partial_i r}{\partial x_i}$$

Example $\nabla \times [f(r)\hat{r}]_x$

$$= \cancel{\varepsilon_{ijk}\,\partial_i f(r)\frac{r_j}{r}} = \frac{\partial}{\partial y}\frac{z f(r)}{r} - \frac{\partial}{\partial z}\frac{y f(r)}{r} = 0$$

$$\cancel{= \varepsilon_{ijk}\left[\frac{[r_j \partial_i f(r) + \delta_{ij} f(r)]\, r - \frac{x_i}{r} f(r)\, r_j}{r^2}\right]}$$

$$= \varepsilon_{ijk}\,\partial_i f(r)\frac{r_j}{r}\vec{e}_k$$

$$= \varepsilon_{ijk}\, r_j \frac{d}{dr}\left(\frac{f(r)}{r}\right)\frac{r_j}{r}\vec{e}_k = 0.$$

$\nabla \times B = 0 \Rightarrow$ irrotational

$$\begin{vmatrix} \hat{r} & \theta & \phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ h_r V_r & h_\theta V_\theta & h_\phi V_\phi \end{vmatrix} \frac{1}{h_r h_\theta h_\phi}$$

4.5 Laplacian

$$\nabla^2 \varphi = \nabla \cdot \nabla \varphi$$

$$\nabla^2 A = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) (A_x, A_y, A_z) = (\nabla \cdot \nabla) \vec{A}$$

Example

$$\nabla^2 \varphi(r) = \nabla \cdot \frac{d\varphi(r)}{dr} \hat{e}_r = \dots$$

$$\nabla^2 r^n = n(n+1) r^{n-2}$$

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(1) $\nabla \times \nabla \varphi = 0$

$= \varepsilon_{ijk} \partial_i \partial_j \varphi \, \vec{e}_k$

$\nabla \cdot (\nabla \times \vec{V}) = 0$

$= \partial_i \varepsilon_{kji} \partial_k V_j$

$= \varepsilon_{kji} \partial_i (\partial_k V_j) = 0.$

Example

$$\begin{cases} \nabla \cdot E = \rho / \varepsilon_0 \\ \nabla \times E = -\frac{\partial B}{\partial t} \\ \nabla \cdot B = 0 \\ \nabla \times B = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} \end{cases}$$

$$\nabla (\nabla \cdot E) - (\nabla \cdot \nabla) \vec{E}$$

$$\nabla \times (\nabla \times E) = -\frac{\partial}{\partial t} (\nabla \times B) = -\mu_0 \frac{\partial \vec{J}}{\partial t} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\Rightarrow \nabla^2 \vec{E} - \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \frac{1}{\varepsilon_0} \nabla \rho + \mu_0 \frac{\partial \vec{j}}{\partial t}.$$

4.6 Indentities!

$$\nabla (\vec{A} \cdot \vec{B}) = (\vec{B} \cdot \nabla) \vec{A} + (\vec{A} \cdot \nabla) \vec{B} + \vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A}).$$

$$\nabla \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \nabla) \vec{A} - (\vec{A} \cdot \nabla) \vec{B} + \vec{A} (\nabla \cdot \vec{B}) - \vec{B} (\vec{\nabla} \cdot \vec{A})$$

(1) $\nabla (A \cdot B)$

$= \nabla_A (A \cdot B) + \nabla_B (A \cdot B)$

$= B \times (\nabla \times A) + (B \cdot \nabla) \vec{A} + (\vec{A} \times \nabla \times \vec{B}) + (\vec{A} \cdot \nabla) \vec{B}$

(2) $\nabla \times (A \times B)$

$= \nabla_A \times (A \times B) + \nabla_B \times (A \times B)$

$= (B \cdot \nabla) \vec{A} - \vec{B} (\nabla \cdot \vec{A}) + - -$

4.7 Integral

Example. $r \to \infty$, $f, A \xrightarrow{\text{strongly}} 0.$

$$\int \vec{A}(\vec{r}) \cdot \nabla f(\vec{r}) \, d^3 \vec{r}$$

$$= \iint dy\, dz \left[A_x \frac{\partial f}{\partial x} dx \right] + \dots$$

$$= \iint dy\, dz \left[A_x f \Big|_{-\infty}^{+\infty} - \int f \frac{\partial A_x}{\partial x} dx \right] + \dots$$

$$= - \iiint dx\, dy\, dz \; f \frac{\partial A_x}{\partial x} - \dots$$

$$= - \int f(r) \, \nabla \cdot \vec{A} \; d^3 \vec{r}$$

$$\int f(\vec{r}) \, \nabla \cdot \vec{A} \, d^3 \vec{r} = - \int \vec{A}(\vec{r}) \cdot \nabla f(\vec{r}) \, d^3 \vec{r}$$

$$\int \vec{C}(\vec{r}) \cdot (\nabla \times \vec{A}(\vec{r})) \, d^3 \vec{r} = \int \vec{A}(\vec{r}) \cdot (\nabla \times \vec{C})$$

4.8. Integral Theorem.

Gauss' Theorem

$$\oint_{\partial V} A \cdot d\sigma = \int_V \nabla \cdot A \, d\tau$$

Corollary : Green's Theorem

identities : $\nabla\cdot(u\nabla v) = u\nabla^2 v + (\nabla u)\cdot(\nabla v)$

$\nabla\cdot(v\nabla u) = v\nabla^2 u + (\nabla u)\cdot(\nabla v)$

1. $\int_V (u\nabla^2 v - v\nabla^2 u)\, d\tau = \oint_{\partial V} (u\nabla v - v\nabla u)\cdot d\vec{\sigma}$

2. $\oint_{\partial V} (u\nabla v)\cdot d\vec{\sigma} = \int_V u\nabla^2 v\, d\tau + \int_V (\nabla u)\cdot(\nabla v)\, d\tau$.

$\vec{B}(x,y,z) = B(x,y,z)\vec{a}$

$\vec{a}\cdot\oint_{\partial V} \vec{B}\, d\sigma = \int_V \nabla\cdot(B\vec{a})\, d\tau = \vec{a}\cdot\int_V \nabla B\, d\tau$

$\Rightarrow a\cdot[\quad] = 0$

$\oint_{\partial V} \vec{B}\, d\sigma = \int_V \nabla B\, d\tau$.

$\vec{B} = \vec{a}\times\vec{P}$

$\oint_{\partial V} (\vec{a}\times\vec{P})\cdot d\vec{\sigma} = \int_V \nabla\cdot(\vec{a}\times\vec{P})\, d\tau$

$= \left[\oint_{\partial V} \vec{P}\times d\sigma\right]\cdot\vec{a} = \int_V -(\nabla\times\vec{P})\cdot\vec{a}\, d\tau$ ✓

$\Rightarrow \oint_{\partial V} d\vec{\sigma}\times\vec{P} = \int_V \nabla\times\vec{P}\, d\tau$

4.9

Stoke's Theorem

$\oint_{\partial S} \vec{B}\cdot d\vec{r} = \int_S \nabla\times\vec{B}\cdot d\vec{\sigma}$

$\int_S d\vec{\sigma}\times\nabla\varphi = \oint_{\partial S} \varphi\, d\vec{r}$

$\int_S (d\vec{\sigma}\times\vec{\nabla})\times\vec{P} = \oint_{\partial S} d\vec{r}\times\vec{P}$

Example $I = \int J\cdot d\sigma = \frac{1}{\mu_0}\int_S \nabla\times B\, d\sigma \Rightarrow$

$\Leftarrow \int_S (\nabla\times E)\, d\sigma = -\frac{d}{dt}\int B\cdot d\sigma = -\frac{d\phi}{dt}$

4.9. Potential Theory.

φ, A.

$\cancel{\nabla\times\nabla\varphi = 0}$

∇

$\nabla\cdot E = \rho/\varepsilon_0$

$\nabla\times E = -\frac{\partial B}{\partial t}$

$\nabla\cdot B = 0 \longrightarrow B = \nabla\times\vec{A}$

$\nabla\times B = \mu_0\vec{j} + \mu_0\varepsilon_0\frac{\partial E}{\partial t}$

$\nabla\times(\vec{E} + \frac{\partial\vec{A}}{\partial t}) = 0 \Rightarrow E = -\nabla\varphi - \frac{\partial\vec{A}}{\partial t}$

Lorentz Gauge : $\frac{1}{c^2}\frac{\partial\varphi}{\partial t} + \nabla\cdot\vec{A} = 0$

$\frac{\rho}{\varepsilon_0} = \nabla\cdot E = -\nabla^2 E - \frac{\partial}{\partial t}\nabla\cdot A = -\nabla^2\varphi + \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}$

$\nabla\times B = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial E}{\partial t}$

$\nabla(\nabla\cdot A) + \nabla^2 A = \mu_0 j + \mu_0\varepsilon_0\frac{\partial}{\partial t}(-\nabla\varphi - \frac{\partial\vec{A}}{\partial t}) \Rightarrow \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - \nabla^2 A = \mu_0\vec{J}$

$$\left\{\begin{array}{l} \nabla^2\varphi = -\rho/\varepsilon_0 \\ \nabla^2\varphi = 0. \\ \nabla^2\varphi = -\frac{q}{\varepsilon_0}\delta(\vec{r}) \end{array}\right.$$

$\nabla^2\varphi = \delta$

4.10. Helmholtz Theorem.

$\nabla\cdot\vec{F}(\vec{r}) = D(\vec{r})$

$\nabla\times\vec{F}(\vec{r}) = \vec{C}(\vec{r})$

则在 $\vec{F}(\vec{r},t) \xrightarrow{|\vec{r}|\to+\infty} \frac{1}{|\vec{r}|^{1+\alpha}} \to 0$, $\alpha > 0$

有 $\vec{F}(\vec{r}) = -\frac{1}{4\pi}\vec{\nabla}\int_{全} d^3r'\frac{D(\vec{r}')}{|\vec{r}-\vec{r}'|} + \frac{1}{4\pi}\vec{\nabla}\times\int_{全} d^3r'\frac{\vec{C}(\vec{r}')}{|\vec{r}-\vec{r}'|}$.

Proof : 对 $\vec{F}(\vec{r})$

令 $\vec{W}(\vec{r}) = \frac{1}{4\pi}\int_{全} d^3r'\frac{\vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|}$，有 (转至最后)

5.1 Curvilinear Coordinate

Orthogonal

$$\vec{V}(\vec{r}) = V_r \hat{e}_r + V_\theta \hat{e}_\theta + V_\varphi \hat{e}_\varphi$$

$$dx = \frac{\partial x}{\partial q_1} dq_1 + \frac{\partial x}{\partial q_2} dq_2 + \frac{\partial x}{\partial q_3} dq_3$$

$$(dr)^2 = dx^2 + dy^2 + dz^2$$
$$= \left(\frac{\partial x}{\partial q_i} dq_i\right)^2 + \dots$$
$$= \left(\frac{\partial x}{\partial q_i}\frac{\partial x}{\partial q_j} dq_i dq_j\right) + \dots$$
$$= g_{ij}\, dq_i dq_j$$

$$g_{ij} = \frac{\partial x}{\partial q_i}\frac{\partial x}{\partial q_j} + \frac{\partial y}{\partial q_i}\frac{\partial y}{\partial q_j} + \dots \quad \text{metric}$$

Specialize

$$(dr)^2 = (h_1 dq_1)^2 + (h_2 dq_2)^2 + (h_3 dq_3)^2$$
$$h_i = \left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial q_i}\right)^2$$
$$\Rightarrow dr_i = h_i dq_i \quad \Rightarrow \quad h_i \hat{e}_i = \frac{\partial \vec{r}}{\partial q_i}$$
$$d\vec{r} = h_1 dq_1 \hat{e}_1 + h_2 dq_2 \hat{e}_2 + h_3 dq_3 \hat{e}_3$$

$$\nabla = \hat{e}_1 \frac{\partial}{h_1 \partial q_1} + \hat{e}_2 \frac{\partial}{h_2 \partial q_2} + \hat{e}_3 \frac{1}{h_3}\frac{\partial}{\partial q_3}$$

$$\nabla \cdot V = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(V_1 h_2 h_3) + \dots \right]$$

$$\nabla \times B = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} \hat{e}_1 h_1 & \hat{e}_2 h_2 & \hat{e}_3 h_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 B_1 & h_2 B_2 & h_3 B_3 \end{vmatrix}$$

Dirac.

$$\begin{cases} E\psi_A - c(\sigma\cdot p)\psi_B = mc^2 \psi_A \\ c\sigma\cdot p\, \psi_A - E\psi_B = mc^2 \psi_B \end{cases}$$

$$\left[\begin{pmatrix} E\mathbb{1}_2 & 0 \\ 0 & -E\mathbb{1}_2 \end{pmatrix} - \begin{pmatrix} 0 & c(\sigma\cdot p) \\ -c(\sigma\cdot p) & 0 \end{pmatrix}\right]\begin{pmatrix}\psi_A \\ \psi_B\end{pmatrix} = mc^2 \begin{pmatrix}\psi_A \\ \psi_B\end{pmatrix}$$

$$[(\sigma_3 \otimes \mathbb{1}_2)E - \gamma \otimes c(\sigma\cdot p)]\Psi = mc^2\Psi$$

~~Re~~ ~~Rew~~ Refine. $\Psi = \begin{pmatrix}\psi_A \\ \psi_B\end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}$, $\gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$

$$\gamma_0 = \sigma_3 \otimes \mathbb{1}_2 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2\end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$\gamma^i = \gamma \otimes \sigma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0\end{pmatrix}$$

$$[\gamma_0 E - c(\gamma_1 p_1 + \gamma_2 p_2 + \gamma_3 p_3)]\psi = mc^2\psi$$

Let $\gamma^0 \gamma^i = \alpha_i$

$$[\gamma^0 mc^2 + c(\alpha_1 p_1 + \dots + \alpha_3 p_3)]\Psi = E\Psi$$

Dirac-equation

$(\gamma^0)^2 = 1 \quad (\gamma^i)^2 = -1$ Clifford algebra.

$\gamma^\mu \gamma^i + \gamma^i \gamma^\mu = 0$

complete basis.

$\mathbb{1}_4$, $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, $\gamma^\mu\ (0\sim 3)$

$\gamma^5\gamma^\mu$. $\sigma^{\mu\nu} = i\gamma^\mu\gamma^\nu$

Helmholtz

$$\nabla^2 \vec{w}(\vec{r}) = \frac{1}{4\pi}\nabla^2 \int_\Omega d^3r' \frac{\vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|} = \frac{1}{4\pi}\int_\Omega d^3r' \vec{F}(\vec{r}')\, \nabla^2 \frac{1}{|\vec{r}-\vec{r}'|} = -\int_\Omega d^3r' \vec{F}(\vec{r}')\, \delta(\vec{r}-\vec{r}') = -\vec{F}(\vec{r})$$

$$\Rightarrow \vec{F}(\vec{r}) = -\nabla^2 \vec{w}(\vec{r}) = -[\vec{\nabla}(\nabla\cdot\vec{w}) - \nabla\times(\nabla\times\vec{w})]$$

$$-\nabla\cdot\vec{w}(\vec{r}) = -\frac{1}{4\pi}\vec{\nabla}\cdot\int_\Omega d^3r' \frac{\vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|} = -\frac{1}{4\pi}\int_\Omega d^3r' \vec{F}(\vec{r}')(-1)\vec{\nabla}' \frac{1}{|\vec{r}-\vec{r}'|}$$
$$= \frac{1}{4\pi}\int d^3r' \left[\nabla'\cdot\frac{\vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|} - \frac{1}{|\vec{r}-\vec{r}'|}\vec{\nabla}'\cdot\vec{F}(\vec{r}')\right]$$
$$= \frac{1}{4\pi}\int d\vec{a}'\cdot\frac{\vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|} - \frac{1}{4\pi}\int_\Omega d^3r' \frac{\vec{\nabla}'\cdot\vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$\nabla\times\vec{w}(\vec{r}) = \nabla\times\frac{1}{4\pi}\int_\Omega d^3r' \frac{\vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|} = -\frac{1}{4\pi}\int d^3r' \left(\nabla'\frac{1}{|\vec{r}-\vec{r}'|}\times\vec{F}(\vec{r}')\right)$$

$$= -\frac{1}{4\pi}\int d^3\vec{r}' \left\{ \vec{\nabla}' \times \left(\frac{\vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|} \right) - \frac{\nabla' \times \vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|} \right\}$$

$$= -\frac{1}{4\pi}\int_{\mathcal{B}} d\vec{a}' \times \frac{\vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|} + \frac{1}{4\pi}\int_{3} d^3 r \, \frac{\nabla' \times \vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|}.$$

Q.E.D

习题，

1. ~~$[J_x, J_x] = i J_z$ (2.2.31)~~

   A. B two noncommuting Hermitian matrices.

   $$AB - BA = iC$$

   Prove C is Hermitian (2.2.36.)

2. (i) $\vec{r}' = U\vec{r}$. U is unitary , r is a vector with complex ,

   show $|r|$ is invariant .

   (ii) $U: \vec{r} \to \vec{r}'$, $r^\dagger r = r'^\dagger r$ $\Rightarrow$ U is unitary . (2.2.51)

3. $\hat{L} = -i(\vec{r} \times \nabla)$,

   (a) $L_x + iL_y = e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta \frac{\partial}{\partial\varphi}\right)$

   (b) $L_x - iL_y = -e^{i\varphi}\left(\frac{\partial}{\partial\theta} - i\cot \frac{\partial}{\partial\varphi}\right)$

   (c) $[L_i, L_j] = i\varepsilon_{ijk}L_k$ (3.10.31)