张量与微分形式

1. Tensor Analysis

1.1. Introduction GR. ED, Material

Scalar – 0

vector – 1 , Dimension , transform

tensor of rank n , d-dimension

→ 1. n indices , $n = 1 \sim d$, d^n component .

2. transformed in a special manner .

components

1.2. Covariant and Contravariant Tensor / (Cartesian)

$$\begin{cases} A_i' = \sum_j (e_i' \cdot \hat{e}_j) A_j = \sum_j \frac{\partial x_i'}{\partial x_j} A_j :: \sum_j \frac{\partial x_j}{\partial x_i'} A_j \\ (\nabla\varphi)_i' = \frac{\partial \varphi}{\partial x_i'} = \sum_j \frac{\partial x_j}{\partial x_i'} \frac{\partial \varphi}{\partial x_j} = \sum_j \frac{\partial x_i'}{\partial x_j} \frac{\partial \varphi}{\partial x_j} \end{cases}$$

$$(A')^i = \frac{\partial x'^i}{\partial x^j} A^j$$ contravariant vector

$$A_i' = \frac{\partial x^j}{\partial x'^i} A_j \qquad (\nabla\varphi)_i' = \frac{\partial \varphi}{\partial x'^i}$$ Covariant vector

1.3. Tensor of rank 2.

$$(A')^{ij} = \frac{\partial x'^i}{\partial x^k} \frac{\partial x'^j}{\partial x^l} A^{kl}$$

$$(A')^i{}_j = \frac{\partial x'^i}{\partial x^k} \frac{\partial x^l}{\partial x'^j} A^k{}_l$$

$$(A')_{ij} = \frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j} A_{kl} .$$

Cartesian coordinates

All the same

$$(A')^{ij} = S_{ik} A^{kl} (S^T)_{lj} \Leftrightarrow A' = S A S^T \quad \Leftrightarrow \cancel{\vec{A}' = \vec{S}\vec{A}\,S}$$

Similarity transformation

~~Con Congreu~~

congruent tra-

1.4. Addition and Subtraction of tensors

$$A + B = C$$

$$A^{ij} + B^{ij} = C^{ij} .$$

Symmetry .

$A^{mn} = A^{nm}$ symmetric

$A^{mn} = -A^{nm}$ anti-

$$A^{mn} = \frac{1}{2}\underbrace{(A^{mn} + A^{nm})}_{s} + \frac{1}{2}\underbrace{(A^{mn} - A^{nm})}_{a}$$

1.5. Isotropic tensor

$$\delta^k{}_l \frac{\partial x'^i}{\partial x^k} \frac{\partial x^l}{\partial x'^j} = \cancel{(\delta^i_j)} \frac{\partial x'^i}{\partial x^k} \cdot \frac{\partial x^k}{\partial x'^j} = \frac{\partial x'^i}{\partial x'^j} = \delta^i{}_j$$

And it is isotropic .

1.6 Contraction

$$\vec{A} \cdot \vec{B} = A^i B_i$$

$$(B')^i{}_i = \frac{\partial x'^i}{\partial x^k} \cdot \frac{\partial x^l}{(\partial x')^i} B^k{}_l = \frac{\partial x^l}{\partial x^k} B^k{}_l = \delta_{lk} B^k_l = B^k_k$$

~~Sca~~ Scalar , trace is invariant .

$$\mathrm{tr}(P^{-1}AP) = \mathrm{tr}(APP^{-1}) = \mathrm{tr}(A)$$

1.7. Direct Product .

$$C^{ij}{}_{klm} = A^i{}_k B^j{}_{lm} , \quad F^{ij}{}_{kl} = A^j B^i{}_{kl}$$

$$(\nabla \vec{E}) = \partial_i E_j$$

$(\vec{A} \cdot \nabla)\vec{B}$

$A_i \partial_i B_j$

Example 1.

$$C'^j_i = a_i' b'^j = \frac{\partial x^k}{\partial x_i'} a_k \frac{\partial x'^j}{\partial x^l} b^l = \frac{\partial x^k}{\partial x_i} \frac{\partial x'^j}{\partial x^l} c^l{}_k$$

[illegible] Tensor

Generally,

$$\frac{\partial x^i}{\partial (x')^j} \neq \left(\frac{\partial (x')^j}{\partial x^i}\right)^{-1}$$

Cartesian system

$$\frac{\partial x^i}{(\partial x')^j} = \frac{(\partial x')^j}{\partial x^i}$$

1.8. Quotient rule.

$$K_i A^i = B$$

?

$$K_{iljk} A^{ik} = B_{kl}$$

?

if equation holds in all transformed coordinate system, then K is a tensor

$$K_i^{\ j} A_j = B_i \longrightarrow (K')_i^{\ j} A_j' = B_i'$$

$$B_i' = \frac{\partial x^m}{\partial (x')^i} B_m = \frac{\partial x^m}{\partial x'^i} K_m^{\ j} A_j$$

$$= \frac{\partial x^m}{\partial x'^i} K_m^{\ j} \frac{\partial x'^n}{\partial x^j} A_n'$$

$$= \frac{\partial x^m}{\partial x'^i} \frac{\partial x'^j}{\partial x^n} K_m^{\ n} A_j' = (K')_i^{\ j} A_j'$$

$$\Rightarrow \left[(K')_i^{\ j} - \frac{\partial x^m}{\partial (x')^i} \frac{(\partial x')^j}{\partial x^n} K_m^{\ n}\right] A_j = 0$$

$x = r\cos\theta \qquad \theta = \arctan \frac{y}{x}$

$y = r\sin\theta$

$\frac{\partial x}{\partial \theta} \cancel{= -r\sin\theta} \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$

$\cancel{\frac{\delta\theta}{\delta}}$

$$\frac{\partial \theta}{\partial y} = \frac{1/x}{1+\frac{y^2}{x^2}} = \frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2}$$

$$\left(\frac{\partial \theta}{\partial y}\right)^{-1} = \frac{r}{\cos\theta}$$

$$\cancel{\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \begin{bmatrix} r \\ \theta \end{bmatrix}}$$

$$\cancel{\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \end{bmatrix} \begin{bmatrix} r \\ \theta \end{bmatrix}}$$

$$\cancel{\begin{bmatrix} r \\ \theta \end{bmatrix} = \begin{bmatrix} & \end{bmatrix}} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

Example. 2

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right] A^\mu = J^\mu$$

Scalar

vector

1.9. Spinor | Spin 0 scalar, 1 vector, 2 tensor, 1/2 → spinor

2.1 Pseudovectors ~~dual tensor~~

$A' = SA \qquad \leftarrow$ vector

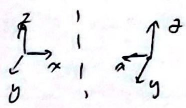
$A' = \det(S)\, SA \qquad \leftarrow$ pseudo vector

$T \otimes T = P \otimes P = T \qquad T \otimes P = P \otimes T = P.$

Example 3. 三阶全反对称不变赝张量

$$\eta_{ijk} = \det(A) \sum_{pqr} a_{ip}\, a_{jq}\, a_{kr}\, \varepsilon_{pqr}$$

← 赝张量.



2.2. Dual Tensor

Anti-symetric second-rank tensor C, associte with (, ,)

a pseudovector

$$\text{Vector} \rightarrow C_i = \frac{1}{2} \varepsilon_{ijk} C^{jk} \qquad (C_1, C_2, C_3) = (C^{23}, C^{31}, C^{12})$$

$$C = \begin{pmatrix} 0 & C^{12} & -C^{31} \\ -C^{12} & 0 & C^{23} \\ C^{31} & -C^{23} & 0 \end{pmatrix}$$

simply different representation of the same thing

$$V^{ijk} = A^i B^j C^k$$

$$V = \varepsilon_{ijk} V^{ijk}$$

$$= \begin{vmatrix} A^1 & B^1 & C^1 \\ A^2 & B^2 & C^2 \\ A^3 & B^3 & C^3 \end{vmatrix}$$

Hodge dual

$$\varepsilon: \underbrace{T_{i_1 \cdots i_p}}_{p} \longmapsto \frac{1}{(n-p)!} \varepsilon_{i_1 \cdots i_n} \underbrace{T_{i_{p+1} \cdots i_n}}_{(n-p)}$$

3. Tensors in general coordinate

3.1 Metric tensor

3.1.1 Covariant basis vector ε_i

$$\hat{\varepsilon}_i = \frac{\partial x}{\partial q^i} \hat{e}_x + \frac{\partial y}{\partial q^i} \hat{e}_y + \frac{\partial z}{\partial q^i} \hat{e}_z$$

$$\vec{A} = A^1\hat{\varepsilon}_1 + A^2\hat{\varepsilon}_2 + A^3\hat{\varepsilon}_3$$

$$ds^2 = \hat{\varepsilon}_i\, dq^i \cdot \hat{\varepsilon}_j\, dq^j$$

$$= \hat{\varepsilon}_i \cdot \hat{\varepsilon}_j\, dq^i dq^j$$

$$\Rightarrow g_{ij} = \hat{\varepsilon}_i \cdot \hat{\varepsilon}_j$$ neither unit nor mutually orthogonal

covariant tensor

$$g^{ik} g_{kj} = g_{jk} g^{ki} = \delta^i_j$$

Use to make connection between contra – co.

$$g_{ij} F^j = F_i \quad , \quad g^{ij} F_j = F^i$$

$$\vec{A} = A^i \hat{\varepsilon}_i = A^i \delta_i^k \hat{\varepsilon}_k = (A^i g_{ij})(g^{jk}\hat{\varepsilon}_k) = A_j \hat{\varepsilon}^j$$

3.1.2 Contra – Co Bases

$$\hat{\varepsilon}^i = \frac{\partial q^i}{\partial x}\hat{e}_x + \cdots$$ ← Contra Basis vector

$$\hat{\varepsilon}^i \cdot \hat{\varepsilon}_j = \frac{\partial q^i}{\partial x}\frac{\partial x}{\partial q^j} + \cdots = \delta^i_j$$

$$(\hat{\varepsilon}^i \cdot \hat{\varepsilon}^j)(\hat{\varepsilon}_j \cdot \hat{\varepsilon}_k) = \delta^i_k$$

$$\left(\frac{\partial q^i}{\partial x}\frac{\partial q^j}{\partial x} + \cdots\right)\left(\frac{\partial x}{\partial q^j}\frac{\partial x}{\partial q^k} + \cdots\right) = \delta^i_k$$

$$\cancel{\frac{\partial q^i}{\partial x^l}\frac{\partial q^j}{\partial x^l}\, g_{jk}}\ \partial x^l$$

$$\Rightarrow g^{ij} g_{jk} = \delta^i_k$$

$$g^{ij} = \hat{\varepsilon}^i \cdot \hat{\varepsilon}^j \rightarrow \boxed{g^{ij}\hat{\varepsilon}_j = \hat{\varepsilon}^i}$$

Example 4.

$x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$

$\varepsilon_r =$

$\varepsilon_\theta =$

$\varepsilon_\varphi =$

$\varepsilon^r =$

$\varepsilon^\theta =$

$\varepsilon^\varphi =$

$\Rightarrow$ $g_{11} =$, $g_{22} =$, $g_{33} =$ $\Rightarrow$

$$g_{ij} = \begin{pmatrix} 1 & & \\ & r^2 & \\ & & r^2\sin^2\theta \end{pmatrix}$$

$$g^{ij} = \begin{pmatrix} 1 & & \\ & r^2 & \\ & & (r\cos\theta)^2 \end{pmatrix}$$

$$\vec{A}\cdot\vec{B} = (A^i\hat{\varepsilon}_i)\cdot(B_j\hat{\varepsilon}^j) = A^i B_j(\varepsilon_i \cdot \varepsilon^j) = A^i B_i$$

$$(\vec{\nabla}\psi)_j = \frac{\partial\psi}{\partial q^i}\frac{\partial q^i}{\partial x_j} = \frac{\partial\psi}{\partial q^i}\hat{\varepsilon}^i = \frac{\partial\psi}{\partial q^i} g^{il}\hat{\varepsilon}_l = \frac{\partial\psi}{\partial q_l}\hat{\varepsilon}_l$$

$$= \frac{\partial\psi}{\partial q_l}\hat{\varepsilon}_l$$

3.2. Covariant Derivatives

$$\vec{r} = x\hat{i} + y\hat{j} = r\hat{r} + 0\,\hat{\theta}$$

$$r = \frac{\partial r}{\partial x}\cdot x + \frac{\partial x}{\partial\theta}\cdot 0$$

$$= \cancel{\varepsilon_r}$$

$$(V')^i = \frac{\partial x^i}{\partial q^k} V^k$$

$$\frac{\partial V'^i}{\partial q^j} = \frac{\partial x^i}{\partial q^k}\frac{\partial V^k}{\partial q^j} + \frac{\partial^2 x^i}{\partial q^j \partial q^k} V^k$$

$$\Rightarrow \frac{\partial \vec{V}'}{\partial q^j} = \frac{\partial V^k}{\partial q^j}\hat{\varepsilon}_k + V_k \frac{\partial \hat{\varepsilon}_k}{\partial q^j}$$

$$\frac{\partial\hat{\varepsilon}_k}{\partial q^j} = \Gamma^\mu_{jk}\hat{\varepsilon}_\mu$$

Γ^μ_{jk} Christoffel Symbols of the 2nd kind

$$\Gamma^m_{jk} = \hat{\varepsilon}^m \cdot \frac{\partial\hat{\varepsilon}_k}{\partial q^j}$$

$$\Gamma^m_{jk} = \Gamma^m_{kj}$$

Proof Refutation

$$\boxed{\frac{\partial\hat{\varepsilon}_1}{\partial q_2} = \frac{\partial\hat{\varepsilon}_2}{\partial q_1}}$$

$$\Rightarrow \frac{\partial V'}{\partial q^j} = \frac{\partial V^k}{\partial q^j}\hat{\varepsilon}_k + V^k \Gamma^\mu_{jk}\hat{\varepsilon}_\mu$$

$$= \left(\frac{\partial V^k}{\partial q^j} + V^\mu \Gamma^k_{j\mu}\right)\hat{\varepsilon}_k$$

$$V^k_{;j} = \frac{\partial V^k}{\partial q^j} + V^\mu \Gamma^k_{j\mu}$$

$$\Rightarrow \frac{\partial V'}{\partial q^j} = V^k_{;j}\,\hat{\varepsilon}_k$$

$\Rightarrow d\vec{V}' = [V_{;}{}^{k}{}_{j}\, dq^j]\, \hat{\varepsilon}_k$ $\quad V_{;}{}^{k}{}_{j}$ is a mixed tensor

Covariant

$\cancel{V_{i;j} = V_{;}{}^{k}{}_{j}\, g_{ik} = \frac{\partial V_i}{\partial q^j} + V_u \Gamma^{u}_{ju\,i} = \frac{\partial V_i}{\partial q^j} - V_k \Gamma^{k}_{ij}}$ ✓

$\cancel{\Gamma^{i}_{ju} = \Gamma^{u}_{ij} \quad \varepsilon^i \cdot \varepsilon_j = \delta^i_j \quad \Gamma^{i}_{jk} \quad \Gamma^{i}_{jk}}$

$\cancel{\frac{\partial(\varepsilon^i \cdot \varepsilon_j)}{\partial q_k} = \varepsilon^i \frac{\partial \varepsilon_j}{\partial q_k} + \varepsilon_j \frac{\partial \varepsilon^i}{\partial q_k} = 0}$

$\frac{\partial V^k}{\partial q^i}$，$\Gamma^{k}_{ju}$ 都不是张量

但 $V_{;}{}^{k}{}_{j}$ 是.

全 括.

$\cancel{\Gamma^{i}_{jk} = \Gamma^{j}_{jk}\ (\ldots)}$

~~(不 学 懂 的 推 导)~~

$\partial_i(U_m U^m) = (\partial_i U_m)U^m + U_m \partial_i U^m$

$(\partial_i U_m) U^m = \partial_i(U_m U^m) - U_m \partial_i U^m$

$(\nabla_i U_m) U^m = \partial_i(U_m U^m) - U_m \nabla_i U^m$

$= \partial_i(U_m U^m) - U_m(\partial_i U^m + \Gamma^{m}_{ij} u^j)$

$= (\partial_i U_m) U^m - \Gamma^{p}_{im} U_p U^m$

$(\nabla_i U_m) U^m = (\partial_i U_m) U^m - \Gamma^{p}_{im} U_p U^m$

$\Rightarrow \nabla_i U_m = \partial_i U_m - \Gamma^{p}_{im} U_p$

球 侧 面

Christoffel symbol of the first kind

$[ij,k] = \cancel{g_{mk}}\ g_{mk} \Gamma^{m}_{ij} = \Gamma_{ij,k}$ ①

$= g_{mk}\, \hat{\varepsilon}^m \cdot \frac{\partial \hat{\varepsilon}_i}{\partial q^j} = \hat{\varepsilon}_k \cdot \frac{\partial \hat{\varepsilon}_i}{\partial q^j}$

$g_{ij} = \hat{\varepsilon}_i \cdot \hat{\varepsilon}_j$

$\frac{\partial g_{ij}}{\partial q^k} = \frac{\partial \hat{\varepsilon}_i}{\partial q^k} \cdot \hat{\varepsilon}_j + \hat{\varepsilon}_i \cdot \frac{\partial \hat{\varepsilon}_j}{\partial q^k} = [ik,j] + [jk,i]$

$\frac{\partial g_{ik}}{\partial q^j} = [ij,k] + [jk,i]$ ← 正交 则有 ~~相等~~ 为 0

归一

$\frac{\partial g_{jk}}{\partial q^i} = [ji,k] + [ki,j]$

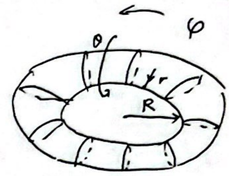
$\frac{1}{2}\left(\frac{\partial g_{ik}}{\partial q^j} + \frac{\partial g_{jk}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^k}\right) = [ij,k]$

① $\Rightarrow g^{nk}[ij,k] = \Gamma^{n}_{ij}$

$\Rightarrow \Gamma^{n}_{ij} = g^{nk}[ij,k] = \frac{1}{2} g^{nk}\left[\frac{\partial g_{ik}}{\partial q^j} + \frac{\partial g_{jk}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^k}\right]$

~~接 下 前~~： Example 1.

甜甜圈上的黎曼几何.



$H_i = \left|\frac{\partial F}{\partial x^i}\right|$

$R,\ r,\ \theta,\ \varphi$

$x = (R + r\cos\theta)\cos\varphi$

$y = (R + r\cos\theta)\sin\varphi$

$z = r\sin\theta$

$\frac{\partial x}{\partial\theta} = -r\sin\theta\cos\varphi \qquad \frac{\partial x}{\partial\varphi} = -(R + r\cos\theta)\sin\varphi$

$\frac{\partial y}{\partial\theta} = -r\sin\theta\sin\varphi \qquad \frac{\partial y}{\partial\varphi} = (R + r\cos\theta)\cos\varphi$

$\frac{\partial z}{\partial\theta} = r\cos\theta \qquad \frac{\partial z}{\partial\varphi} = 0$

$H_\theta = r,\ H_\phi = R + r\cos\theta$

$g_{\theta\theta} = r^2,\ g_{\phi\phi} = (R + r\cos\theta)^2,\ g_{\theta\phi} = g_{\phi\theta} = 0$

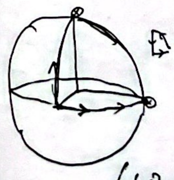
$g_{\mu\nu} = \begin{bmatrix} r^2 & \\ & (R + r\cos\theta)^2 \end{bmatrix} \qquad dV = \sqrt{|g|}\, dx_1 \cdots dx_n.$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} H_\theta & 0 \\ 0 & H_\phi \end{bmatrix} \begin{bmatrix} e_\theta \\ e_\varphi \end{bmatrix}$ ✓

$\frac{\partial(\sqrt{F_3})}{\partial x_2}$ —

1. Levi-Civita 平移

与测地线夹角不变.



$A_{\mu;\nu} = \frac{\partial A_\mu}{\partial x^\nu} - \Gamma^{\alpha}_{\mu\nu} A_\alpha$

$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}\left(g_{\mu\beta,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}\right).$

$\Gamma^{\alpha}_{\mu\nu} = \begin{pmatrix} (0,0), & (0,\ ) \\ (\ \ ), & (\ \ ) \end{pmatrix}$ $\quad \Gamma^{1}_{11} = \Gamma^{1}_{12} = 0$

$\Gamma^{1}_{21} = 0,\ \Gamma^{1}_{22} = \frac{(R + r\cos\theta)\sin\theta}{r}$

Γ^2_{11} Γ^2_{12} Γ^2_{21} $\Gamma^2_{22} = 0$

$= 0$ $=$ $= \sim$

$$\cancel{\tfrac{1}{2} g^{21} (g_{11,2} + g_{21,1} - g_{12,1})} + \tfrac{1}{2} g^{22} (\cancel{g_{12,2}} + g_{22,1} \cancel{+ g_{12,2}})$$

$$= \frac{1}{2} \frac{1}{(R + r\cos\theta)^2} \cdot 2(R + r\cos\theta) \cdot -\sin\theta \cdot r \checkmark.$$

矢量平移

$$\nabla_\nu A_\mu = \partial_\nu A_\mu \mp \Gamma^\alpha_{\mu\nu} A_\alpha = 0$$

$$\partial_\nu A_\mu = \Gamma^\alpha_{\mu\nu} A_\alpha$$

~~球 矢量梯度~~ 球坐标

$\cancel{(\nabla \vec{A})_{ij} = \frac{\partial A_i}{\partial q^i}}$

$x = r\sin\theta\cos\varphi$

$y = r\sin\theta\sin\varphi$

$z = r\cos\theta$

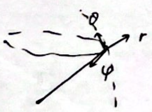
$\frac{\partial x}{\partial r} = \sin\theta\cos\varphi$ $\frac{\partial x}{\partial \theta} = r\cos\varphi\cos\theta$ $\frac{\partial x}{\partial \varphi} = -r\sin\theta\sin\varphi$.

$\frac{\partial y}{\partial r} = \sin\theta\sin\varphi$ $\frac{\partial y}{\partial \theta} = r\sin\varphi\cos\theta$ $\frac{\partial y}{\partial \varphi} = r\sin\theta\cos\varphi$

$\frac{\partial z}{\partial r} = \cos\theta$ $\frac{\partial z}{\partial \theta} = -r\sin\theta$ $\frac{\partial z}{\partial \varphi} = 0.$

$H_r = 1$ $H_\theta = r$ $H_\varphi = r\sin\theta$

$$g_{\mu\nu} = \begin{array}{c|ccc} & r & \theta & \varphi \\ \hline r & 1 & 0 & 0 \\ \theta & 0 & r^2 & 0 \\ \varphi & 0 & 0 & r^2\sin^2\theta \end{array}$$



$\cancel{\Gamma^r_{rr} = 0 \quad \Gamma^r_{r\theta} = 0} \quad \Gamma^r_{r\varphi} = 0$ $\Gamma^\alpha_{\beta\sigma}$ 27个

$\cancel{\Gamma^r_{\theta r} = 0 \quad \Gamma}$

$\cancel{\Gamma^r_{rr} = 0 \quad \Gamma^r_{r\theta} = 0 \quad \Gamma^r_{r\varphi} = 0}$

$\cancel{\Gamma^r_{\theta r} = 0 \quad \Gamma^r_{\theta\theta} = -r \quad \Gamma^r_{\theta\varphi} = 0}$

$\cancel{\Gamma^r_{\varphi r} = 0 \quad \Gamma^r_{\varphi\theta} = 0 \quad \Gamma^r_{\varphi\varphi}} =$

$\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta}$ 削去9个，剩18个.

$$\cdot\ \Gamma^\alpha_{r\alpha} = \frac{1}{2} g^{\alpha\beta} (\cancel{\partial g}\, g_{r\beta,\alpha} + g_{\alpha\beta,r} - g_{r\alpha,\beta})$$

$$= \frac{1}{2} g^{\alpha\alpha} (\cancel{g_{r\beta,\alpha}}\, \cancel{g_{r\alpha,\alpha}} + g_{\alpha\alpha,r} \cancel{- g_{r\alpha,\alpha}})$$

$$= \frac{1}{2} g^{\alpha\alpha} g_{\alpha\alpha,r}$$

$\cdot\ \cancel{\Gamma^1_{01} = \tfrac{1}{2}}$ $\Gamma^r_{rr} = \frac{1}{2} g^{rr} g_{rr,r} = 0$

$\cdot\ \Gamma^\theta_{r\theta} = \frac{1}{2} g^{\theta\theta} g_{\theta\theta,r} = r \cdot / r^2 = 1/r$

$\cdot\ \Gamma^\varphi_{r\varphi} = \frac{1}{2} g^{\varphi\varphi} g_{\varphi\varphi,r} = 2 r\sin^2\theta / r^2\sin^2\theta = 1/r$

剩15个

Γ^i_{rj} $(i, j \neq r,\ i \neq j)$

$$\Gamma^i_{rj} = \frac{1}{2} g^{i\beta} (g_{r\beta,j} + g_{j\beta,r} \mp g_{rj,\beta})$$

$$= \frac{1}{2} g^{ii} (g_{ri,j} + g_{ji,r} - g_{rj,i})$$

$$= 0$$

剩13个.

$\varepsilon_r = (\sin\theta\cos\theta,\ \sin\theta\sin\varphi,\ \cos\theta)$

$\varepsilon_\theta = r(\cancel{\sin\theta\sin}\ \cos\theta \overset{\cos\varphi}{\cancel{\sin\varphi}},\ \sin\varphi\cos\theta,\ -\sin\theta)$

$\varepsilon_\varphi = +r(-\sin\theta\sin\varphi,\ \sin\theta\,\cancel{\cos\varphi})\ 0\ )$

$\varepsilon^r = g^{r\beta} \varepsilon_\beta = \varepsilon_r = \ldots$

$\varepsilon^\theta = g^{\theta\beta} \varepsilon_\beta = \frac{1}{r}($ $)$

$\varepsilon^\varphi = g^{\varphi\varphi} \varepsilon_\varphi = \frac{1}{r\sin\theta}($ $)$

$$\Gamma^{\alpha}_{\beta\sigma} = \begin{array}{c} \alpha \\ \theta \\ \\ \end{array}\left[\begin{array}{ccc} [0,0,0] & [0,-R,0] & [0,0,\ -R\cos^2\theta] \\ [0,1/R,0] & [1/R,0,0] & [0,0,\cos\theta\sin\theta] \\ [0,0,1/R] & [0,0,\tan\theta] & [1/R,-\tan\theta,0] \end{array}\right] \quad (x)$$

$$(\nabla \vec{A}) \quad \begin{array}{c|ccc|} & A_\rho & A_\theta & A_\varphi \\ \frac{\partial}{\partial\rho} & \frac{\partial A_\rho}{\partial\rho}+0 & \frac{\partial A_\theta}{\partial\rho}+0 & \frac{\partial A_\varphi}{\partial\rho}+0 \\ \frac{\partial}{\partial\theta} & \frac{\partial}{\rho\partial\theta}A_\rho & \frac{\partial}{\rho\partial\theta}A_\theta+\frac{A_\rho}{\rho} & \frac{\partial}{\rho\partial\theta}A_\varphi \\ \frac{\partial}{\partial\varphi} & \frac{\partial}{\rho\sin\theta}\frac{\partial A_\rho}{\partial\varphi} & \frac{1}{\rho\sin\theta}\frac{\partial A_\theta}{\partial\varphi} & \frac{1}{\rho\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}+\frac{A_\rho}{\rho} \end{array}$$

$$\nabla_\mu A_\nu = \frac{\partial A_\nu}{\partial q^\mu} - A_\rho \Gamma^{\rho}_{\mu\nu}$$

$$\begin{array}{cccc} & A_1 & A_2 & A_3 \\ \frac{\partial}{\partial q^1} & 0 & 0 & \rho \\ \frac{\partial}{\partial q^2} & 0 & \theta & \rho \\ \frac{\partial}{\partial q^3} & 0 & 0 & \theta \end{array}$$

- Tensor Derivative Operator

Gradient

$$\nabla\psi = \frac{\partial\psi}{\partial q^i}\hat{\varepsilon}^i$$

Divergence

$$\nabla\cdot\vec{V} = \hat{\varepsilon}^j\cdot\frac{\partial(V^i\hat{\varepsilon}_i)}{\partial q^j} = \hat{\varepsilon}^j\cdot\left(\frac{\partial V^i}{\partial q^j} + \Gamma^i_{jk}V_k\right)\hat{\varepsilon}_i = \frac{\partial V^i}{\partial q^i} +$$

$$\Gamma^i_{ik} = \frac{1}{2}g^{im}\left[\frac{\partial g_{im}}{\partial q^k} \cancel{+\frac{\partial g_{km}}{\partial q^i} - \frac{\partial g_{ik}}{\partial q^m}}\right] = \frac{1}{2}g^{im}\frac{\partial g_{im}}{\partial q^k}\ V^k\ \Gamma^i_{ik}$$

$$\frac{d(\det(g))}{dq^k} = \det(g)\, g^{im}\frac{\partial g_{im}}{\partial q^k}$$

$$\Gamma^i_{ik} = \frac{1}{2\det(g)}\frac{d\det(g)}{dq^k} = \frac{1}{\sqrt{|g|}}\frac{\partial\sqrt{|g|}}{\partial q^k}$$

$$\Rightarrow \nabla\cdot\vec{V} = V^i_{;i} = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial q^k}\left(\sqrt{|g|}\,V^k\right).$$

Laplacian

$$g^{ki}\frac{\partial\psi}{\partial q^i} \longrightarrow \nabla\cdot\vec{V}$$

$$\nabla^2\psi = \frac{}{[} \quad - - - \quad]$$

$$= \frac{1}{h_1h_2h_3}\frac{\partial}{\partial q^i}\left(\frac{h_1h_2h_3}{h_i^2}\cdot\frac{\partial\psi}{\partial q^i}\right)$$

Curl

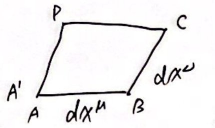
$$\frac{\partial V_i}{\partial q^j} - \frac{\partial V_j}{\partial q^i} = V_{i,j} - V_k\Gamma^k_{ij} - V_{j,i} - V_k\Gamma^k_{ji}$$

$$= V_{i;j} - V_{j;i}$$

Jacobian

$$\begin{pmatrix} \frac{ds(u_1)}{du_1} \\ \vdots \\ \frac{ds(u_n)}{} \end{pmatrix} = \begin{pmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \frac{\partial x_3}{\partial u_3} \\ & & \\ & J & \end{pmatrix}\begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \vdots \end{pmatrix} \quad J^{-1}$$

Introduction to GR



$$dV = V_A - V_{A'}$$

$$= [(V_C - V_D) - (V_B - V_A)] - [(V_C - V_B) - (V_D - V_{A'})]$$

$$= [\nabla_\nu dx^\nu \nabla_\mu dx^\mu V - \nabla_\mu dx^\mu \nabla_\nu dx^\nu V]$$

$$= dx^\mu dx^\nu V [\nabla_\nu, \nabla_\mu]$$

$$[\nabla_\nu, \nabla_\mu]$$

$$= (\partial_\nu + \Gamma_\nu)(\partial_\mu + \Gamma_\mu) - (\partial_\mu + \Gamma_\mu)(\partial_\nu + \Gamma_\nu)$$

$$= (\partial_\nu \partial_\mu + \Gamma_\nu \partial_\mu + \partial_\nu \Gamma_\mu + \Gamma_\nu \Gamma_\mu) -$$

$$(\partial_\mu \partial_\nu + \partial_\mu \Gamma_\nu + \Gamma_\mu \partial_\nu + \Gamma_\mu \Gamma_\nu)$$

$$[\tfrac{\partial}{\partial x^\mu}, F] V$$

$$= \frac{\partial}{\partial x^\mu} F V - F \frac{\partial}{\partial x^\mu} V$$

$$= (\frac{\partial}{\partial x^\mu} F) V.$$

$$= -[\partial_\mu, \Gamma_\nu] + [\partial_\nu, \Gamma_\mu] + [\Gamma_\nu, \Gamma_\mu]$$

$$= -\frac{\partial \Gamma_\nu}{\partial x^\mu} + \frac{\partial \Gamma_\mu}{\partial x^\nu} + [\Gamma_\nu, \Gamma_\mu] = R_{\mu\nu}$$

$$\cancel{R = g^{\mu\nu} R_{\mu\nu}}$$

$$\cancel{\nabla_\rho R = \nabla_\rho (g^{\mu\nu}) R_{\mu\nu} + g^{\mu\nu} \nabla_\rho R_{\mu\nu}}$$

$$\cancel{= g^{\mu\nu} \nabla_\rho (R_{\mu\nu})}$$

$$\cancel{g_{\mu\nu} \nabla_\rho R = g}$$

$$\cancel{= g_{\rho\sigma} g^{\mu\nu} \nabla_\rho (R_{\mu\nu}) = g_{\rho\sigma} \nabla_\rho R = \nabla_\rho R\, g_{\rho\sigma}}$$

$$\nabla_\tau \frac{dx^\mu}{d\tau} = 0$$

$$\Rightarrow \frac{\partial}{\partial \tau} \frac{\partial x^\mu}{\partial \tau} + \Gamma = 0$$

$$\frac{\partial^2 x^\mu}{\partial \tau} = -\Gamma \approx F$$

$$\Gamma = \frac{1}{2} g^{\sigma d} (\quad \cdots \quad)$$

$$= \frac{1}{2} \frac{\partial g_{00}}{\partial x} \approx \Gamma = -\frac{\partial \phi}{\partial x}$$

$$g_{00} \approx 2\phi + c$$

$$\nabla^2 \phi = 4\pi G \rho$$

$$\nabla^2 (g_{00}) = 8\pi G \rho$$

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$R_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\nabla T_{\mu\nu} = 0$$

$$\nabla R_{\mu\nu} \neq 0$$

But $\nabla_\nu (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) = 0$

$$\nabla_\nu (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu}) = 0$$

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G T_{\mu\nu}}{c^4}}$$

$$[\nabla_\mu, [\nabla_\nu, \nabla_\kappa]] + [\nabla_\nu, [\nabla_\kappa, \nabla_\mu]] + [\nabla_\kappa, [\nabla_\mu, \nabla_\nu]] = 0$$

Bianchi Identity

$$R_{abmn;l} + R_{ablm;n} + R_{abnl;m} = 0$$

$$g^{bn} g^{am} (\qquad) = 0$$

$$g^{bn} (R^m{}_{bmn;l} - R^m{}_{bml;n} + R^m{}_{bnl;m}) = 0$$

$$R^n{}_{n;l} - R^n{}_{l;n} - R^{nm}{}_{nl;m} = 0$$

$$R_{;l} - R^n{}_{l;n} - R^m{}_{l;m} = 0$$

$$R_{;l} = 2R^m{}_{l;m}$$

$$\Rightarrow \nabla_l R^l{}_m = \frac{1}{2} \nabla_m R.$$

Differential forms

1. Introduction $dx \; dy \; dz$

$w = A\,dx + B\,dy + C\,dz$ 1-form

$w = F\,dx\wedge dy + G\,dx\wedge dz + H\,dy\wedge dz$ 2-form

$w = K\,dx\wedge dy\,dz$ 3-form

exterior algebra (Grassmann Algebra)

p-form, p factors dx_i, no dx_i, 0-form

2. Exterior algebra

$((aw_1)+(bw_2))\wedge w_3 = aw_1\wedge w_3 + bw_2\wedge w_3 \quad (p_1 = p_2)$

$(w_1\wedge w_2)\wedge w_3 = w_1\wedge(w_2\wedge w_3)$

$a(w_1\wedge w_2) = (aw_1)\wedge w_2$

$dx_i\wedge dx_j = -dx_j\wedge dx_i$

$a\,dx_1\wedge b\,dx_2 = -a(b\,dx_2\wedge dx_1) = -ab(dx_2\wedge dx_1) = ab\,dx_1\wedge dx_2$

$\varepsilon_p\, dx_{h_1}\wedge dx_{h_2}\wedge\cdots\wedge dx_{h_p} \quad 1\le h_1 \le h_2\cdots < h_p.$

$\boxed{p\le d}$

Example.

$w = (3dx+4dy-dz)\wedge(dx-dy+2dz)$

$= 7(dy\wedge dz - dz\wedge dx - dx\wedge dy)$

Complementary (Dual) DF.

d, p-form $\quad$ (d-p)-form $\quad$ Hodge star operater

Metric / orientation. [Orthogonal]

$*w$ 1. (indices of w) followed by (indices of w')

2. $(-1)^{\mu}$ (Metric tensor diagonal element)

3D - Euclidean.

$*1 = dx_1\wedge dx_2\wedge dx_3$

$*dx_1 = dx_2\wedge dx_3$ - - -

$*(dx_1\wedge dx_2) = dx_3$ ———

$*(dx_1\wedge dx_2\wedge dx_3) = 1$

$\boxed{*(*w) = w}$

Example 4D - Minkowski

$$\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$*1 = dt\wedge dx\wedge dy\wedge dz$

$*(dt\wedge dx\wedge dy\wedge dz) = -1$

$*dx_1 = dt\wedge dx_2\wedge dx_3$, $\quad *dx_i = dt\wedge dx_j\wedge dx_k$

$*(dt\wedge dx_1) = -dx_2\wedge dx_3$, $\quad *(dt\wedge dx_i) = -dx_j\wedge dx_k$

$A = A_x dx + A_y dy + A_z dz$

$B = B_x dx + B_y dy + B_z dz$

$\Rightarrow A\wedge B = (A_yB_z - A_zB_y)\,dy\,dz + \cdots$

$*(A\wedge B) = (A_yB_z - A_zB_y)\,dx + \cdots$

$= (\vec{A}\times\vec{B})_x\,dx + \cdots$

$A\wedge B\wedge C = (A_xB_yC_z - \cdots)\,dx\wedge dy\wedge dz$

$*(A\wedge B\wedge C) = \vec{A}\cdot(\vec{B}\times\vec{C})$

Exterior Derivatives $\quad P = p'$ $\quad \cancel{w = w'}$

$$d(w+w') = dw + dw'$$
$$d(fw) = (df)\wedge w + f\,dw$$
$$d(w\wedge w') = dw\wedge w' + (-1)^p w\wedge dw'$$
$$d(dw) = 0$$
$$df = \sum_j \frac{\partial f}{\partial x_j} dx_j$$

Example.

$$w = A\, dx_1 \wedge \cdots \wedge dx_p \qquad dw = \frac{\partial A}{\partial x_\mu} dx_\mu \wedge \cdots$$
$$w' = B\, dx_1 \wedge \cdots \wedge dx_{p'} \qquad dw' = \frac{\partial B}{\partial x_\mu} dx_\mu \wedge \cdots$$

$$d(w\wedge w') = d(AB)[dx_1\wedge\cdots\wedge dx_p]\wedge[dx_1\wedge\cdots\wedge dx_{p'}]$$
$$= \sum_\mu \left[\frac{\partial A}{\partial x_\mu} B + A\frac{\partial B}{\partial x_\mu}\right] dx_\mu \wedge [\ -\ —\ ]\wedge[\ —\ ]$$
$$= dw\wedge w' + (-1)^p w\wedge dw'.$$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$
$$d(df) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) dx\wedge dx + \cdots$$
$$+ \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) dy\wedge dx + \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) dx\wedge dy = 0$$

~~Define~~ $\Rightarrow df = (\nabla f)_x dx + \cdots$

$$dw = d(A_x dx + A_y dy + A_z dz) \qquad (1)$$
$$= \left(\frac{\partial A_x}{\partial x} dx + \frac{\partial A_x}{\partial y} dy + \frac{\partial A_x}{\partial z} dz\right)\wedge dx + \cdots$$
$$= \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right] dy\wedge dz + \cdots$$
$$= (\nabla\times A)_x\, dy\wedge dz.$$

$$* d(A_x dx + B_y dy + A_z dz) = (\nabla\times A)_x dx + \cdots$$

$$d(B_x dy\wedge dz + B_y dz\wedge dx + B_z dx\wedge dy)$$
$$= \left(\frac{\partial B_x}{\partial x} dx + \cdots\right)\wedge dy\wedge dz + \cdots$$
$$= \left[\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right) dx\wedge dy\wedge dz$$
$$\Rightarrow *d(\ —\ —\ ) = \nabla\cdot\vec{B} \qquad (2)$$

---

in (1) Let $w = df$

$$d(df) = (\nabla\times\nabla f)_x\, dy\wedge dz + \cdots = 0.$$

in (2). $dB = d(A_x dx + \cdots)$

$$d(d(A_x dx + \cdots)) = \nabla\cdot(\nabla\times A)\, dx\wedge dy\wedge dz = 0$$

Example.

$$F_{\mu\nu} = \begin{array}{c} \\ t \\ x \\ y \\ z \end{array}\begin{array}{c} \begin{array}{cccc} t & x & y & z \end{array} \\ \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix} \end{array}$$

$$F = -E_x\, dt\wedge dx - E_y\, dt\wedge dy - E_z\, dt\wedge dz$$
$$+ B_x\, dy\wedge dz + B_y\, dz\wedge dx + B_z\, dx\, dy$$

$$J = \rho\, dx\wedge dy\wedge dz - J_x\, dt\wedge dy\wedge dz - J_y\, dt\wedge dz\wedge dx - J_z\, dt\wedge dx\wedge dy$$

$dF = 0 \Rightarrow$ | $dJ = 0 \Rightarrow$ (Bianchi Identi). | $d(*F) = J$ ✓

$*F = \cdots$

$d(*F) = J \Rightarrow$

Integrating forms.

1- form

$$\int_c w = \int_c [A_x dx + A_y A_y] = \int_{t_P}^{t_Q} [A_x(t) \frac{dx}{dt} + A_y(t) \frac{dy}{dt}] dt$$

If it's independent about $P \sim Q$.

w is exact, $w = df(x, y)$

设 $w = df$, $w = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$

$\Rightarrow \int_P^Q w = f(Q) - f(P)$

$$\int_S w = \int_S B(x,y)\, dx \wedge dy$$

$x = au + bv$, $y = eu + fv$

$dx \wedge dy = \cdots (af - be) du \wedge dv$

Stoke's Theorem

Simply connected region R of p-d differentiable manifold in n-d space.

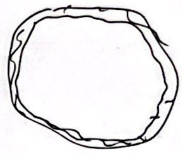
R has a boundary ∂R. d is $p-1$

w is a $(p-1)$ form defined on R and boundary.

with derivative dw

then $$\int_R dw = \int_{\partial R} w$$

$w = A\, dx_2 \wedge \cdots dx_p$



$dw = \frac{\partial A}{\partial x_1} dx_1 \wedge dx_2 \wedge \cdots dx_p$

$$\int_\Delta dw = \int_{\partial \Delta} \int_{x_1 - \delta}^{x_1} \left(\frac{\partial A}{\partial x_1}\right) dx_1 \wedge \cdots \wedge dx_p$$

$$= \int_{\partial \Delta} A(x_1, \cdots) dx_2 \wedge \cdots dx_p - \int_{\partial \Delta} A(x_1 - \delta) dx_2 \wedge \cdots \wedge dx_p.$$

✓

Example. Green's Theorem in Plane

$w = P\, dx + Q\, dy$

$dw = \frac{\partial P}{\partial y} dy \wedge dx + \frac{\partial Q}{\partial x} dx \wedge dy$

$= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy$

$\Rightarrow \int_C (P dx + Q\, dy) = \int_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) ds$

$W = A_x dx + A_y dy + A_z dz$.

$dw =$ ______

$\int_C A_x dx + \cdots = \int_C \vec{A} \cdot d\vec{r} = \int_S (\nabla \times A) \cdot d\vec{\sigma}$.

$w \sim E_x dy \wedge dz + E_y dz \wedge dx + \cdots$

$dw = \left[\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \right] dx \wedge dy \wedge dz$

$\Rightarrow \int (\nabla \cdot E) d\tau = \int_{\partial V} \vec{E} \cdot d\vec{\sigma}$