又因为 系数分解唯一

$\because\ V = m \oplus m'$ 且 $\dim m' = \cancel{0-m}$ dimO − dim

---

$\forall\, |\psi\rangle \in V^p(\mathbb{F})$

$|\psi\rangle = \alpha_1 |v_1\rangle + \cdots + \alpha_0 |v_0\rangle$

$= \alpha_1 |v_1\rangle + \cdots + \alpha_n |v_n\rangle \cancel{+}$ — — ——

$V^p(\mathbb{F}) = V_1^n(\mathbb{F}) + V_2^{p-n}(\mathbb{F})$

$= (m \oplus m') \oplus V_2 = m \oplus m' \oplus V_2$ — ——

最多拆到1

即每个基底

投影

已知 m 为 V 的一子空间

当且仅当 $\forall |v\rangle \in V$，均有 $\hat{P}|v\rangle = |u\rangle$

其中 $|v\rangle = \underset{\in m}{|u\rangle} + \underset{\in m'}{|u'\rangle}$

$\left.\right\}$ m 与 m' 垂直 后再说明.

$V = \cancel{m + m'}\ m \oplus m'$

$\hat{P}$ 为 V 中一算符. (OP)

则称 $\hat{P}$ 为一将 V 映射到子空间 M 之投影，记为 $\hat{P}_{V\to M}$ ($\hat{P}_m$)

可以证明：投影为一 LO.

考察 $\hat{P}(\alpha|v\rangle + \beta|w\rangle) \overset{?}{=} \alpha\hat{P}|v\rangle + \beta\hat{P}|w\rangle$

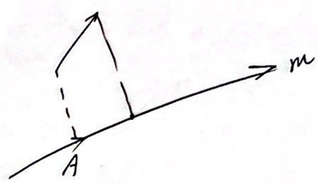
$= \hat{P}\big(\alpha(|u\rangle + |u'\rangle) + \beta(|\eta\rangle + |\eta'\rangle)\big)$

$= \hat{P}\big((\alpha|u\rangle + \beta|\eta\rangle + \alpha|u'\rangle + \beta|\eta'\rangle)\big)$ ✓

$= \alpha|u\rangle + \beta|\eta\rangle = \alpha\hat{P}|v\rangle + \beta\hat{P}|w\rangle$.

[定理] 若 $\hat{P}$ 为一投影

则有 $\hat{P}^2 = \hat{P}$



证明：$\forall |v\rangle \in V$

$\because\ V = \cancel{m}\cdot m \oplus m'$

$|v\rangle = |u\rangle + |u'\rangle$

$\therefore\ \hat{P}^2|v\rangle = \hat{P}^2(|u\rangle + |u'\rangle)$

$= \hat{P}\cdot\hat{P}(|u\rangle + |u'\rangle)$

$= \hat{P}(|u\rangle)$

$= \hat{P}(|u\rangle + |0\rangle)$

$= \cancel{\hat{P}}\,|u\rangle = \hat{P}|v\rangle$.

[定理] 若 $\hat{P}_{V\to m}$，则 $\hat{P}' = \hat{1} - \hat{P}$ 为 $V \to m'$ 之投影.

证明：$V = m \oplus m'$

有 $|v\rangle = \underset{\in m}{|u\rangle} + \underset{\in m'}{|u'\rangle}$

$\hat{P}'|v\rangle = \hat{P}'(|u\rangle + |u'\rangle)$

$= (\hat{1} - \hat{P})(|u\rangle + |u'\rangle)$

$= \hat{1}(u$ ———— $)$

$= |u\rangle + |u'\rangle - |u\rangle = \cancel{\hat{P}}\,|u'\rangle$.

特例.

① $\hat{1}: V \to V$

② $\hat{0}: V \longrightarrow \{|0\rangle\}$

则 $V = V \oplus \{|0\rangle\}$ $\quad \hat{0} = \hat{1} - \hat{1}$.

$V^D = M^n \oplus M^{(D-n)}$ 正交直和.

$\hat{I}_{|\psi\rangle} = \hat{P}_{|\psi\rangle} + \hat{P}_{|\psi\rangle}$

$|\psi\rangle = \underline{c_1|v_1\rangle + \cdots + c_n|v_n\rangle} + \underline{c_{n+1}|v_{n+1}\rangle + \cdots + c_D|v_D\rangle}$

可以看出 投影算符~~通常~~不可逆,
唯一可逆的投影是 $\hat{I}$.

$\hat{A}$ 是一 LO, 有 $\hat{A}|v_1\rangle = \lambda_1|v_1\rangle$ $\quad |v_1\rangle \neq |v_2\rangle \neq 0$
$\qquad A|v_2\rangle = \lambda_2|v_2\rangle$

① 若 $\lambda_1 \neq \lambda_2$, 则 $|v_1\rangle$ 和 $|v_2\rangle$ LI

② $V_1 = \{\alpha_1|v_1\rangle \mid \alpha_1 \in F\}$ 为 V^D 一子空间.

③ V_1 为一在 $\hat{A}$ 作用下不变子空间.

④ $D=2$, $V = V_1 \oplus V_2$, 其中 $V_2 = \{\alpha_2|v_2\rangle \mid \alpha_2 \in F\}$.

---

① 反证 $a_1|v_1\rangle + a_2|v_2\rangle = 0$

$|v_1\rangle = -\frac{a_2}{a_1}|v_2\rangle$

$-\cancel{\frac{a_2}{a_1}}\hat{A}|v_2\rangle = \lambda_1 \cdot \cancel{-\frac{a_2}{a_1}}|v_2\rangle$

$\lambda_1 = \lambda_2$ 矛盾.

② ✓

③ $\alpha_1\hat{A}|v_1\rangle = \alpha_1\lambda_1|v_1\rangle$ ✓

④ $V = V_1 + V_2$, $|w\rangle \in V_1 \cap V_2$.

$|w\rangle = \alpha_1|v_1\rangle = \alpha_2|v_2\rangle$

$\alpha_1 = \alpha_2 = 0 \Rightarrow$ ✓ 不变 ✓

定义 V
线性独立
$\sum$ 坐标
子空间 $\longrightarrow$ 直和)分解
表示: 唯一 $|\psi\rangle = \sum_i \alpha_i|v_i\rangle$
分解唯一.

内积 $\hat{F}$

是纯数.

定义 $V(F)$ 中两个 $|v\rangle$, $|w\rangle$ 之内积, 记作 $\langle v|w\rangle$

且 ① $\langle w|v\rangle = \langle v|w\rangle^*$ $(\in \mathbb{C})$

② $\langle v|v\rangle \geq 0$, $|v\rangle = |0\rangle$ 取.

线性 ③ $\langle v|(\alpha|f\rangle + \beta|g\rangle) = \alpha\langle v|f\rangle + \beta\langle v|g\rangle$

①, ③ $\Rightarrow$ $\langle \alpha f + \beta g|v\rangle$
$= \alpha^*\langle f|v\rangle + \beta^*\langle g|v\rangle$

内积空间: 有向量内积之定义的向量空间.

关于 $\langle v|$: $\langle v| \leftrightarrow |v\rangle$

$\langle v|w\rangle = (|v\rangle, |w\rangle)$

$\langle v| = (|v\rangle, \_)$

$\langle v|\hat{A} = (|v\rangle, \hat{A}\_)$

$\langle v|\alpha f + \beta g\rangle = (|v\rangle, \alpha|f\rangle + \beta|g\rangle)$

$\langle \alpha f + \beta g|v\rangle = (\alpha|f\rangle + \beta|g\rangle, |v\rangle)$

$|v\rangle\langle w| = |v\rangle(|w\rangle, -)$

正交. 内积为 0, 正交

长度 $\|v\| = (\langle v|v\rangle)^{\frac{1}{2}}$

iff $\|v\| = 0$, $|v\rangle = 0$

单位向量 $\|v\| = 1$

性质 1. $\langle v|0\rangle = 0$

2. $\langle v|f\rangle = 0 \quad \forall |v\rangle$
$|f\rangle = 0$

3. iff $\langle v|f\rangle = \langle v|g\rangle \quad \forall |v\rangle$
$|f\rangle = |g\rangle$

定理：两两相互正交的向量组，必定 LI.

$\alpha_1|v_1\rangle + \cdots + \alpha_n|v_n\rangle = 0$

左乘 $\langle v_i|$ ⇒ $\cdots = 0$.

归一正交基底

Gram-Schmidt 正交化.

$|\psi\rangle = \langle e_1|\psi\rangle|e_1\rangle + \cdots + \langle e_n|\psi\rangle|e_n\rangle$

$= |e_1\rangle\langle e_1|\psi\rangle + \cdots |e_n\rangle\langle e_n|\psi\rangle$

$= (|e_1\rangle\langle e_1| + \cdots + |e_n\rangle\langle e_n|)|\psi\rangle$

$= \hat{1}|\psi\rangle$

可以反过来

归一正交基底的完备性

$|e_1\rangle\langle e_1| + \cdots + |e_n\rangle\langle e_n| = \hat{1}$

$(\vec{e}_1\vec{e}_1 + \cdots + \vec{e}_n\vec{e}_n = \vec{\vec{1}})$

非归一正交

$$\begin{pmatrix} \langle v_1|\psi\rangle \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} \langle v_1|v_1\rangle & \langle v_1|v_2\rangle & \cdots & \langle v_1|v_n\rangle \\ & & \langle v_i|v_j\rangle & \\ & & & \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ \vdots \end{pmatrix}$$

$c_i = V^{-1}_{ij}\langle v_j|\psi\rangle$.

$\langle v_i|v_j\rangle\langle v_k|v_j\rangle$

$= \cancel{\delta_{ij}\delta_{kj} = \delta_{ik}}$.

~~正交 ⇒ 有正~~

$= m\delta_{ij}\, n\delta_{ik}$

$= mn\,\delta_{ik}$

$\Rightarrow |\psi\rangle = c_i|v_i\rangle = V^{-1}_{ij}\langle v_j|\psi\rangle|v_i\rangle$

$= (V^{-1}_{ij}|v_i\rangle\langle v_j|)|\psi\rangle$

$\Rightarrow V^{-1}_{ij}|v_i\rangle\langle v_j| = \hat{1}$.

$\langle v|w\rangle$ 内积 $(|v\rangle, |w\rangle)$　$|v\rangle\langle w|$ 线性算符 $|v\rangle(|w\rangle, -)$

三个不等式

~~定理 $\dim V = N_\perp$~~ 定理 $\dim V = N_\perp$

$\langle\ ,\ \rangle$　$\dim V = n$

不可数 $N_\perp$ LI　→ Schmidt.

Schwarz 不等式

$\forall |v\rangle\ |w\rangle$.

必有 $|\langle v|w\rangle| \le \|v\|\|w\|$　"=" 线性相关成立

证：$|w\rangle = 0$，成立

$|w\rangle \ne 0$

令 $|z\rangle = |v\rangle - \frac{\langle w|v\rangle}{\|w\|^2}|w\rangle$

$\langle z|z\rangle = \langle v|v\rangle + \frac{\langle w|v\rangle^*}{\|w\|}\frac{\langle w|v\rangle}{\|w\|}\langle w|w\rangle$

$- \frac{\langle w|v\rangle^*}{\|w\|^2}\langle w|v\rangle - \frac{|\langle w|v\rangle|^2}{\|w\|}$

$= \|v\|^2 - \frac{|\langle w|v\rangle|}{\|w\|^2} = \|z\|^2 \ge 0$.　证完

三角不等式　$\forall |v\rangle, |w\rangle$,

$\|v+w\| \le \|v\| + \|w\|$　"="　线性相关　$\mathrm{Im}=0$　$\mathrm{Re}>0$

证：$|v\rangle = 0$ or $|w\rangle = 0$ ✓

$\|v+w\|^2 = \langle v+w|v+w\rangle = \langle v|v\rangle + \langle w|w\rangle + \langle w|v\rangle + \langle v|w\rangle$

$= \|v\|^2 + \|w\|^2 + 2\,\mathrm{Re}\langle v|w\rangle$

$\le \|v\|^2 + \|w\|^2 + 2|\langle v|w\rangle|$

$\le (\|w\| + \|v\|)^2$ ✓

Bessel 不等式

$|e_1\rangle \cdots |e_m\rangle$ 为内积空间 $V(F)$ 中的一组归一正交向量，

α $|v\rangle$ 为任一向量

$$\|v\|^2 \geq \sum_{i=1}^{m} |c_i|^2$$

其中 $c_i \equiv \langle e_i | v \rangle$

且 $|v'\rangle \equiv |v\rangle - \sum_{i=1}^{m} c_i |e_i\rangle$ 与 $|e_1\rangle \cdots |e_m\rangle$ 正交．

证：

$$0 \leq \langle v' | v' \rangle = \langle v | v \rangle + \cancel{c_i c_j} \delta \; c_i^* c_j \langle e_i | e_j \rangle$$

$$- c_i \langle v | e_i \rangle - c_i^* \langle e_i | v \rangle$$

$$= \|v\|^2 + c_i^* c_i - c_i c_i^* - c_i^* c_i$$

$$= \|v\|^2 - \sum |c_i|^2 = \|v'\|^2 \geq 0$$

$\langle e_i | v' \rangle$

$= \langle e_i | v - c_j e_j \rangle$

$= \langle e_i | v \rangle - c_j \langle e_i | e_j \rangle$

$= \langle e_i | v \rangle - c_i$

$= 0.$

习题：① $\langle v | \hat{O} | w \rangle = 0$

② $\langle v | \hat{A} | w \rangle = 0 \;\; \forall |v\rangle, |w\rangle \Rightarrow \hat{A} = \hat{0}$

③ $\langle v | \hat{A} | w \rangle = \langle v | \hat{B} | w \rangle \;\; \forall$.

$\Rightarrow \hat{A} = \hat{B}$

线性算符的厄密共轭 转化．

$\hat{A} \rightarrow V(F)$

设 $\exists \hat{B}$ 在 $V(F)$

$\forall |v\rangle, |w\rangle$,

$$\langle w | \hat{B} | v \rangle = \langle v | \hat{A} | w \rangle^*$$

则 $\hat{B}$ 是 $\hat{A}$ 之厄米共轭　$\hat{B} = \hat{A}^\dagger$．

$\Rightarrow \underbrace{\langle w | \hat{A}^\dagger | v \rangle} = \langle v | \hat{A} | w \rangle^*$，$\forall |v\rangle, |w\rangle \in V(F)$

$(|w\rangle, \hat{A}^\dagger |v\rangle) = (\hat{A}|w\rangle, |v\rangle)$

$\quad //$

$(\hat{A}^\dagger |v\rangle, |w\rangle)^*$

$\Rightarrow \underline{\langle A^\dagger v | w \rangle} = \underline{\langle v | \hat{A} | w \rangle}^*$

证 $(\hat{A}^\dagger)^\dagger = \hat{A}$

证．$\langle v | \hat{A} | w \rangle$

$= \langle w | \hat{A}^\dagger | v \rangle^*$

$= \langle (\hat{A}^\dagger)^\dagger w | v \rangle^*$

$= \langle v | (\hat{A}^\dagger)^\dagger | w \rangle$　□．

Properties.

$(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger \hat{A}^\dagger$

$(\hat{A} + \hat{B})^\dagger = \hat{A}^\dagger + \hat{B}^\dagger$

$(|f\rangle\langle g|)^\dagger = |g\rangle\langle f|$

可以证明 $\hat{A}^\dagger$ 亦为 L.O.

$\langle v | \hat{A}^\dagger (\alpha f + \beta g) \rangle$

$= \langle \alpha f + \beta g | \hat{A} | v \rangle^*$

$= \langle \hat{A} v | \alpha f + \beta g \rangle$

$= \alpha \langle \hat{A} v | f \rangle + \beta \langle \hat{A} v | g \rangle$

$= \alpha \langle v | \hat{A}^\dagger f \rangle + \beta \langle v | \hat{A}^\dagger g \rangle$

$=$ ______

$\hat{A}$ 可以不线性吗？

✓

线性厄米算符.

$\hat{A}^\dagger = \hat{A}$ — —

① $\langle v|\hat{A}|v\rangle =$ 实数

② $\hat{A}+\hat{B}$ 仍为厄米

③ 若 $[\hat{A},\hat{B}]=0$，$\hat{A}\cdot\hat{B}$ 是厄米

①
$\langle v|\hat{A}|v\rangle$
$= \langle v|\hat{A}^\dagger|v\rangle^* \Rightarrow$ ✓

②
$(\hat{A}+\hat{B})^\dagger$
$= \hat{A}^\dagger + \hat{B}^\dagger$
$= \hat{A}+\hat{B}$

③ $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\cdot\hat{A}^\dagger = \hat{B}\hat{A} \overset{\text{对易}}{=} \hat{A}\hat{B}$.

线性幺正算符

$\hat{A}^\dagger = \hat{A}^{-1}$

① $\hat{U}_1 \cdot \hat{U}_2$ 仍为幺正

② $\langle \underline{\hat{U} f} | \underline{\hat{U} g}\rangle = \langle f|g\rangle$

① $(\hat{U}_1\cdot\hat{U}_2)^\dagger = (\hat{U}_2^\dagger\hat{U}_1^\dagger) = \hat{U}_2^{-1}\hat{U}_1^{-1} = (\hat{U}_1\hat{U}_2)^{-1}$

② $(\hat{U}|f\rangle, \hat{U}|g\rangle)$
$= \langle f|\hat{U}^\dagger\hat{U} g\rangle$
$= \langle f|g\rangle$.

① $U^\dagger$ 亦为幺正

内积空间的正交互补性

[定理] m 为内积 V 之一子空间， $m \perp m^\perp$.

令 $m^\perp = \{|v\rangle \mid |v\rangle \in V$ 且 $\langle v|v'\rangle = 0, |v'\rangle \in m\}$.

则 ① $m^\perp$ 为 V 之一子空间

② $m \oplus m^\perp = V$. 正交互补 正交直和

证: ① $m^\perp \subset V$，显然是子空间.

② 令 $|v\rangle \in m \cap m^\perp$

$\therefore \langle v|v\rangle = 0$

$\Rightarrow |v\rangle = 0$

$\therefore m\cap m^\perp = \{|0\rangle\}$

Consider $\forall\, |\psi\rangle \in V$

令 $|e_1\rangle, \cdots |e_m\rangle$ 为 m 中正交归一. $m = \dim m$.

令 $|f\rangle \equiv \alpha_i |e_i\rangle$

$\alpha_i = \langle e_i|\psi\rangle$

令 $|g\rangle = |\psi\rangle - |f\rangle = |\psi\rangle - \alpha_i|e_i\rangle$

$\langle g|f\rangle = \langle \psi|f\rangle - \langle f|f\rangle$

$= \alpha_i\langle\psi|e_i\rangle - \alpha_i\alpha_i^*\langle e_j|e_i\rangle$

$= 0$

故 $|g\rangle \in m^\perp$.

$\therefore \underset{V}{|\psi\rangle} = \underset{m}{|f\rangle} + \underset{m^\perp}{|g\rangle}$.

可以限 $V^3(R)$ 为例.

因此，可以
$V(H) = m_1 \oplus m_2 \oplus \cdots \oplus m_n$

投影算符具有正交性

$m = \{\alpha \frac{\vec{A}}{|\vec{A}|} \mid \alpha \in \mathbb{R}\}$

$m' = m$ 的正补子空间

为什么可以？

① $\langle f| \quad |g\rangle$

② $\langle v| \quad |v\rangle$

任意性

证: $\hat{P}^2 = \hat{P} = \hat{P}^\dagger$, $\hat{P}$ 投影

$\Rightarrow \hat{P}_{V\to m} : V = m \oplus m'$

$\langle v|\hat{P}|v\rangle = \langle v|u\rangle = \langle u+u'|u\rangle$

$= \langle u|u'\rangle + \langle u'|u\rangle = \langle u|u\rangle + 0 = \|u\|^2 \in \mathbb{R}$.

$\langle v|\hat{P}^\dagger|v\rangle = \langle \hat{P}v|v\rangle = \langle v|\hat{P}|v\rangle^* = \langle v|\hat{P}|v\rangle$.

$\Rightarrow \hat{P}^\dagger = \hat{P}$

$\Leftarrow \quad \hat{P}^2 = \hat{P} = \hat{P}^\dagger$

令 $m = \{|u\rangle \mid \hat{P}|u\rangle = |u\rangle, |u\rangle \in V\}$

m 是 V 的一个子空间.

根据正交互补定理，必有 $V = m \oplus m^\perp$，其中 $m^\perp \perp m$

$\forall |v\rangle \in V$，均有

$|v\rangle = \underset{m}{|u\rangle} + \underset{m^\perp}{|u_\perp\rangle}$ 唯一

有 $\hat{P}|v\rangle = \hat{P}(|u\rangle + |u_\perp\rangle) = \hat{P}|u\rangle + \hat{P}|u_\perp\rangle = |u\rangle + 0$?

又 $\because \langle \hat{P}u_\perp|\hat{P}u_\perp\rangle = \langle u_\perp|\hat{P}^\dagger\hat{P}u_\perp\rangle = \langle u_\perp|\hat{P}^2|u_\perp\rangle = \langle u_\perp|\hat{P}|u_\perp\rangle$
$= 0$

又 $\because \hat{P}^2|u_\perp\rangle = \hat{P}|u_\perp\rangle$

$\Rightarrow \hat{P}|u_\perp\rangle \in m$

$\therefore \langle u_\perp|\hat{P}u_\perp\rangle = 0$

$\therefore \hat{P}$ 为 $V \to m$ 投影

② → ①

令 $|\psi\rangle = |\psi_1\rangle + |\psi_2\rangle$

$|\psi'\rangle = |\psi_1\rangle + i|\psi_2\rangle$

$\therefore$ 左 $=\!=$ $\cancel{\langle\psi_1|\hat{B}|\psi_1\rangle} + \cancel{\langle\psi_2|\hat{B}|\psi_2\rangle} + \langle\psi_1|\hat{B}|\psi_2\rangle + \langle\psi_2|\hat{B}|\psi_1\rangle$

$= \langle\psi|\hat{A}|\psi\rangle$

$= \langle\psi_1+\psi_2|\hat{A}|\psi_1+\psi_2\rangle$

$= \cancel{\langle\psi_1|\hat{A}|\psi_1\rangle} + \cancel{\langle\psi_2|\hat{A}|\psi_2\rangle}$

$+ \langle\psi_1|\hat{A}|\psi_2\rangle + \langle\psi_2|\hat{A}|\psi_1\rangle$

$\langle\psi'|\hat{A}|\psi'\rangle =\!= \langle\psi'|\hat{B}|\psi'\rangle$

$= \cancel{\langle\psi_1|\hat{A}|\psi_1\rangle} + \cancel{\langle\psi_2|\hat{A}|\psi_2\rangle}$

$+ i\langle\psi_1|\hat{A}|\psi_2\rangle - i\langle\psi_2|\hat{A}|\psi_1\rangle \qquad i\langle\psi_1|\hat{B}|\psi_2\rangle - i\langle\psi_2|\hat{B}|\psi_1\rangle$

$\Rightarrow \langle\psi_1|\hat{A}|\psi_2\rangle = \langle\psi_1|\hat{B}|\psi_2\rangle$.

相互正交的正交投影

$\hat{P}_1\hat{P}_2 = \hat{0}$，则相互正交.

$\hat{P}_1\hat{P}_2 = 0$

$\Rightarrow (\hat{P}_1\hat{P})^\dagger = \hat{0}^\dagger = \hat{0}$
$\quad\; \overset{\shortparallel}{\hat{P}_2\hat{P}_1}$

定理：若 $\hat{P}_1, \hat{P}_2$ 相互正交，则 $\hat{P}_1: V \to m_1$，$\hat{P}_2: V \to m_2$，之

$m_1 \perp m_2$

证:

$\Rightarrow \begin{cases} \hat{P}_1|v_1\rangle = |v_1\rangle, & |v_1\rangle \in m_1 \\ \hat{P}_2|v_2\rangle = |v_2\rangle, & |v_2\rangle \in m_2 \end{cases}$

$\langle v_1|v_2\rangle = \langle \hat{P}_1v_1|\hat{P}_2v_2\rangle = \langle v_1|\hat{P}_1^\dagger\hat{P}_2|v_2\rangle = \langle v_1|\hat{P}_1\hat{P}_2|v_2\rangle = 0$

$\Leftarrow$

$m_1 \perp m_2$, $\forall |v\rangle \in V$

$|v\rangle = |v_1\rangle + |v_1'\rangle$

$= |v_2\rangle + |v_2'\rangle$

且 $\hat{P}_1|v\rangle = |v_1\rangle$

$\hat{P}_2|v\rangle = |v_2\rangle$

$\therefore \langle v_1|v_2\rangle = 0$

$\langle \hat{P}_1 v_1 | \hat{P}_2 v_2\rangle = \langle v|\hat{P}_1^\dagger \hat{P}_2|v\rangle = \langle v|\hat{P}_1 \hat{P}_2 v\rangle = 0$

$\therefore \hat{P}_1 \hat{P}_2 = 0$.

$V = m \oplus m'$

$\hat{P}: V \to m$, $\hat{1} - \hat{P} = V \to m'$

$|\psi\rangle = |v\rangle + |v'\rangle \Rightarrow (\hat{P} + \hat{1} - \hat{P})|\psi\rangle$

$= \hat{P}|\psi\rangle + (\hat{1} - \hat{P})|\psi\rangle$

$\hat{1} = \hat{P} + \hat{1} - \hat{P}$

$= |v\rangle + |v'\rangle$

$(\hat{1} - \hat{P})$ 是投影. $(\hat{1}-\hat{P})^2 = (\hat{1} - \hat{P}) = (\hat{1}-\hat{P})^\dagger$.

~~且~~ $(\hat{1} - \hat{P})$ 与 $\hat{P}$ 正交., $(\hat{1}-\hat{P})(\hat{P}) = 0$

故 $m \perp m'$.

注记 $\hat{P}_{V\to m}$ $\hat{1} - \hat{P}: V \to m^\perp$, $m^\perp$ 与 m 之正交互补子空间.

$\hat{P}_1, \hat{P}_2$ 均投影, 若 $\hat{P}_1, \hat{P}_2$ 正交, 则 $\hat{P}_1 + \hat{P}_2$ 投影

$\Rightarrow \hat{P}_1 \cdot \hat{P}_2 = \hat{P}_2 \cdot \hat{P}_1 = 0$ | $(\hat{P}_1 + \hat{P}_2)^\dagger = \hat{P}_1 + \hat{P}_2$.

$(\hat{P}_1+\hat{P}_2)^2 = \hat{P}_1^2 + \hat{P}_1\cdot\hat{P}_2 + \hat{P}_2 \cdot\hat{P}_1 + \hat{P}_2^2$

$= \hat{P}_1 + \hat{P}_2$

$\Leftarrow$

$(\hat{P}_1+\hat{P}_2)^2 = \hat{P}_1 + \hat{P}_2$

$= \hat{P}_1 + \hat{P}_1\hat{P}_2 + \hat{P}_2 \hat{P}_1 + \hat{P}_2$

$\Rightarrow \hat{P}_1 \hat{P}_2 + \hat{P}_2 \hat{P}_1 = 0$

$\Rightarrow \hat{P}_1(\hat{P}_1\hat{P}_2 + \hat{P}_2 \hat{P}_1) = \hat{P}_1 \cdot \hat{0} = \hat{0}$ Lo

$= \hat{P}_1 \hat{P}_2 + \hat{P}_1 \hat{P}_2 \hat{P}_1 = \hat{0}$

$\Rightarrow \hat{P}_1\hat{P}_2 = -\hat{P}_1 \hat{P}_2 \hat{P}_1$

$(P_1P_2 + P_2P_1)P_1 = \hat{0}$

$P_1P_2P_1 + P_2P_1 = 0$

$\Rightarrow \hat{P}_2\hat{P}_1 = -\hat{P}_1 \hat{P}_2 \hat{P}_1$

$P_1P_2 = P_2P_1 = \hat{0}$.

$\hat{P}_1 \cdots \hat{P}_n$ 为投影, 若 $\hat{P}_1, \dots, P_n$ 两两正交, 则 $\hat{P}_1 + \cdots + \hat{P}_n$ 投影.

证: $\Rightarrow$ $(P_1+\cdots+P_n)^2 = \sum_{i,j} \hat{P}_i \hat{P}_j = \hat{P}_i \hat{P}_i = P_1 + \cdots + P_n$.

$(P_1+\cdots+P_n)^\dagger = -$ ——

$\Leftarrow$ $P_i^2 = \hat{P}_i = \hat{P}_i^\dagger$

用 Bessel 不等式

$\forall |v\rangle \in V(F)$, 有 $\||v\rangle\|^2 = \langle v|v\rangle \geq \langle \hat{P} v|\hat{P} v\rangle$

$= \langle v|P^\dagger P|v\rangle = \langle v|\hat{P}|v\rangle$

$= \langle v|(\hat{P}_1+\cdots+\hat{P}_n)|v\rangle = \sum_{i=1}^n \langle v|\hat{P}_i|v\rangle = \sum_{i=1}^n \langle v|\hat{P}_i^\dagger \hat{P}_i|v\rangle$

$= \sum \langle \hat{P}_i v|\hat{P}_i v\rangle$

令 $|v\rangle = \hat{P}_k|\psi\rangle$, 有 $\langle \hat{P}_k\psi|\hat{P}_k\psi\rangle \geq \sum_i \langle \hat{P}_i \hat{P}_k \psi|\hat{P}_i\hat{P}_k\psi\rangle$

$= \langle \hat{P}_k^2\psi|\hat{P}_k^2\psi\rangle + \sum_{i\neq k}\langle \hat{P}_i\hat{P}_k\psi|\hat{P}_i\hat{P}_k\psi\rangle$

$\Rightarrow \sum_{i=1, i\neq k}^n \langle \hat{P}_i \hat{P}_k\psi|\hat{P}_i\hat{P}_k\psi\rangle$ | $\Rightarrow \hat{P}_i \hat{P}_k = \hat{0}$.

$\Rightarrow \hat{P}_i \hat{P}_k|\psi\rangle = 0$

在 D维内积空间中 $\{|e_i\rangle\}$　$\langle e_i|e_j\rangle = \delta_{ij}$

$|e_1\rangle\langle e_1|$

……

$|e_D\rangle\langle e_D|$

两两正交，故 $|e_1\rangle\langle e_1| + \cdots + |e_D\rangle\langle e_D|$ 也为正交投影

$V^D(\mathbb{F}) = m \oplus m^\perp$

$= V_1'(\mathbb{F}) \oplus V_2'(\mathbb{F}) \oplus \cdots \oplus V_D'(\mathbb{F})$

$|\psi\rangle = c_1|e_1\rangle + \cdots + c_D|e_D\rangle$

$\hat{1} = \hat{P}_1 + \hat{P}_2 + \cdots + \hat{P}_D$

$|e_1\rangle\langle e_1| \quad \cdots \quad |e_D\rangle\langle e_D|$

$\hat{P}_i : V^D(\mathbb{F}) \to V_i'(\mathbb{F}) = \{c_i|e_i\rangle \mid c_i \in \mathbb{F}\}$.

表示：相对于一组基底

$|\psi\rangle \xrightarrow{\{|v_i\rangle\}} \begin{pmatrix} c_1 \\ \vdots \\ c_D \end{pmatrix}$

$\hat{A} \xrightarrow{\{|v_i\rangle\}} \begin{pmatrix} A_{11} & \cdots & A_{D1} \\ \vdots & \ddots & \\ A_{1D} & & A_{DD} \end{pmatrix}$

$\begin{cases} \hat{A}|v_1\rangle = A_{11}|v_1\rangle + \cdots + A_{D1}|v_D\rangle \\ \end{cases}$

$\hat{A}\hat{B}|f\rangle = |g\rangle \Rightarrow \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix} = \begin{pmatrix} AB \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$

线性算符的本征值定理

$\hat{A} \in \mathcal{L}_0$，$\lambda$是$\hat{A}$的eV，iff 则$\hat{A}-\lambda\hat{1}$不可逆

证：⇐ 有"$\hat{T}$ iff $\hat{T}|v\rangle = |0\rangle$ 唯一解是$|v\rangle=0$，$\hat{T}$可逆"，

$\hat{T}$不可逆，

$(\hat{A}-\lambda\hat{1})|v\rangle = |0\rangle$

$\Downarrow$

$\hat{A}|v\rangle = \lambda|v\rangle$，$|v\rangle \neq |0\rangle$

λ是$\hat{A}$的一eV

⇒ 有$|v\rangle \neq 0$

$\hat{A}|v\rangle = \lambda|v\rangle$

$(\hat{A}-\lambda\hat{1})|v\rangle = |0\rangle$，同定理 $\hat{A}-\lambda\hat{1}$不可逆

本征值谱　（计算方法）

线性厄米算符的本征值谱定理

[定理一]　$\hat{A}$ - 是线厄

① $\hat{A}$的 eV $\in \mathbb{R}$

② 不同λ的$|v_i\rangle$正交

证：① $\hat{A}|v\rangle = \lambda|v\rangle$

左 ⇒ $\langle v|\hat{A}|v\rangle = \lambda\langle v|v\rangle$

右 ⇒ $\langle \hat{A}^\dagger v|v\rangle = \langle \lambda v|v\rangle = \lambda^*\langle v|v\rangle$

⇒ $\langle v|\hat{A}^\dagger|v\rangle = \langle v|\hat{A}|v\rangle = \lambda\langle v|v\rangle$

⇒ $(\lambda-\lambda^*)\||v\rangle\|^2 = 0$

$\lambda = \lambda^*$

② 正交

$\hat{A}|v_1\rangle = \lambda_1|v_1\rangle \Rightarrow \langle v_2|\hat{A}|v_1\rangle = \lambda_1\langle v_2|v_1\rangle$

$\hat{A}|v_2\rangle = \lambda_2|v_2\rangle \Rightarrow \langle \hat{A}v_2|v_1\rangle = \lambda_2^*\langle v_2|v_1\rangle = \lambda_2\langle v_2|v_1\rangle$

$\qquad\qquad\qquad\qquad \langle v_2|\hat{A}|v_1\rangle = \lambda_1\langle v_2|v_1\rangle =$

简并

[定理二] $V^n(\mathbb{F})$（至少）有一组基底由 $\hat{A}$ 的（可归一正交化）本征向量组成

（D 取无限亦可）← 当成基本公设

先找 $V^n(\mathbb{F})$ 某组基底，找本征值 λ_1

$\hat{A} - \lambda_1\hat{I}$ 不可逆，故 $\exists\ |v_1\rangle \neq 0 \in V^n(\mathbb{F})$

s.t. $(\hat{A} - \lambda_1\hat{I})|v_1\rangle = 0$

令 $m_1 = \{\alpha_1|v_1\rangle \mid \alpha_1 \in \mathbb{F}\}$

$V^n = m_1 \oplus m_1^{\perp}$

对于 $\forall |u\rangle \in m_1^{\perp}$，$\langle v_1|\hat{A}|u\rangle = \langle \hat{A}^{\dagger}v_1|u\rangle = \langle \lambda_1 v_1|u\rangle = 0$

☆ ∴ $m_1^{\perp}$ 为 $\hat{A}$ 之作用下的不变子空间.

∴ 重新再记 ……

$\lambda_1 \quad \lambda_2 \quad \cdots \quad \lambda_n$

$|v_1\rangle \quad |v_2\rangle \quad \cdots \quad |v_n\rangle$

即使有简并，亦可 Schmidt 正交化

$\forall \hat{A}$，

$\hat{A} = \hat{I}\cdot\hat{A}\cdot\hat{I}$

$= |e_i\rangle\langle e_i|\hat{A}|e_j\rangle\langle e_j|$

$= A_{ij}|e_i\rangle\langle e_j|$

可以证明，厄米算符 O，在正交归一基底中的表示，一定是厄米矩阵.

任一 N 阶厄米阵一定可对角化

$A \longrightarrow \hat{A}$ 的在归一正交表示

$\hat{A} \longrightarrow \begin{matrix}\lambda_1 & \lambda_2 & \cdots & \lambda_N \\ |v_1\rangle & |v_2\rangle & \cdots & |v_N\rangle\end{matrix}$ 正交可归一化

$A \longrightarrow A' = S^{-1}AS = S^{\dagger}AS$

其中 $S = \Big(\big(|v_1\rangle\big)\big(|v_2\rangle\big)\cdots\big(|v_N\rangle\big)\Big)$

$AS = \big(\langle e_i|A|e_j\rangle\big)\big(|v_1\rangle, \cdots |v_N\rangle\big) = \big(\lambda_1|v_1\rangle, \cdots, \lambda_N|v_N\rangle\big)$

$S^{\dagger}AS = \begin{pmatrix} |v_1\rangle^* \\ |v_2\rangle^* \\ \vdots \\ |v_N\rangle^* \end{pmatrix}\big(\lambda_1|v_1\rangle \cdots \lambda_N|v_N\rangle\big) = \begin{pmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N\end{pmatrix}$

相互对易的厄米 Op

$\hat{A}, \hat{B}$，若 $[\hat{A},\hat{B}] = 0$，则 $\hat{A},\hat{B}$ 至少有一组共同的本征向量组

$\Leftarrow$ $\hat{A}|v_i\rangle = a_i|v_i\rangle$

$\hat{B}|v_i\rangle = b_i|v_i\rangle$

$\forall|\psi\rangle$，$\hat{A}\hat{B}|\psi\rangle = \hat{A}(\hat{B}|\psi\rangle) = \hat{A}(\hat{B}\sum c_i|v_i\rangle)$

$= \hat{A}\sum b_i c_i|v_i\rangle = \sum a_i b_i c_i|\psi\rangle$

$= \hat{B}\hat{A}|\psi\rangle$.

$\Rightarrow$ $\hat{A}\hat{B} = \hat{B}\hat{A}$，$|v\rangle \neq 0$

$\hat{A}|v\rangle = \lambda|v\rangle$

∵ $\hat{A}(\hat{B}|v\rangle) = \hat{A}\hat{B}|v\rangle = \hat{B}(\lambda|v\rangle) = \lambda\hat{B}|v\rangle = \hat{A}(\hat{B}|v\rangle)$

情况一：无简并：

$\hat{B}|v\rangle$ 与 $|v\rangle$ 在同一方向

$\hat{B}|v\rangle = \delta|v\rangle$，故有相同向量组.

情况二、有简并，且设简并度为 2

$$\begin{cases} \hat{A}|v_1\rangle = \lambda|v_1\rangle \\ \hat{A}|v_2\rangle = \lambda|v_2\rangle \end{cases} \rightarrow |v_1\rangle, |v_2\rangle \text{ 归一正交化.}$$

可得 $\hat{B}|v_1\rangle$，$\hat{B}|v_2\rangle = c_{12}|v_1\rangle + c_{22}|v_2\rangle$

$= c_{11}|v_1\rangle + c_{21}|v_2\rangle$

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} \langle v_1|\hat{B}|v_1\rangle & \langle v_1|\hat{B}|v_2\rangle \\ \langle v_2|\hat{B}|v_1\rangle & \langle v_2|\hat{B}|v\rangle \end{pmatrix} = \text{厄米阵}$$

$$\begin{pmatrix} \langle v_1|\hat{B}|v_1\rangle^* & - \\ \langle v_1|\hat{B}|v_2\rangle^* & -- \end{pmatrix} \checkmark$$

↓

可被对角化，~~证完~~

$$\begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}$$

$|v_1\rangle, |v_2\rangle \longrightarrow |u_1\rangle, |u_2\rangle$（又是 $|v_1\rangle, |v_2\rangle$ 线性组合）

$$\begin{cases} \hat{B}|u_1\rangle = b_1|u_1\rangle \\ \hat{B}|u_2\rangle = b_2|u_2\rangle \end{cases}$$

故 $\hat{A}|u_1\rangle = \lambda_1|u_1\rangle$
$\hat{A}|u_2\rangle = \lambda_2|u_2\rangle$ - -

$\hat{A}\ \hat{B}$ 有共同本征向量

证完

~~可以证明，若有 $\hat{A}|v\rangle = \lambda|v\rangle$~~

~~$\Rightarrow f(\hat{A})|v\rangle = f(\lambda)|v\rangle$~~

以上这一套可以用在无限维向量（函数上）

$n \to \infty$

在基底 $|x_n\rangle$ 中

$|f_n\rangle = \langle x_i|f_n\rangle|x_i\rangle = f(x_i)|x_i\rangle$

内积 $\langle f_N|g_N\rangle = \langle f_N\underbrace{|x_i\rangle\langle x_i|}_{1}g_N\rangle = f^*(x_i)\,g(x_i)$ 发散

厚 $V^N(F) \longrightarrow N \to \infty$

归一正交

$\{|x_i\rangle\}$

$\langle x_i|x_j\rangle = \delta_{ij}$

$|x_i\rangle\langle x_i| = \hat{1}$

$\{|\tilde{x}_i\rangle\}$，其中 $|\tilde{x}_i\rangle = \frac{|x_i\rangle}{\sqrt{\Delta x}}$ （归 $\frac{1}{\sqrt{\Delta x}}$ 正交）

$\langle \tilde{x}_i|\tilde{x}_j\rangle = \frac{\delta_{ij}}{\Delta x}$

$\sum \Delta x |\tilde{x}_i\rangle\langle\tilde{x}_i| = \hat{1}$

$|\tilde{f}_N\rangle = \sqrt{\Delta x}|f_N\rangle$

$= \sum_i \Delta x\, f(x_i)|\tilde{x}_i'\rangle$

$f(x_i) = \langle x_i|f_N\rangle = \langle\tilde{x}_i|\tilde{f}_N\rangle = f(x_i)$ （不变了）

$\langle\tilde{f}_N|\tilde{g}_N\rangle = \sum_i \Delta x\, f^*(x_i)\,g(x_i) = \int_a^b dx\, f^*(x)\,g(x)$

$\|f_N\|^2 = \int_a^b dx\,|f(x)|^2$

取 $+\infty$

$\longrightarrow$ 归 $\sqrt{\delta(0)}$ 正交，$\langle x|x'\rangle = \delta(x-x')$

无限维的内积空间

$\int dx\,|x\rangle\langle x| = \hat{1}$.

$|f\rangle \xrightarrow{\{|x\rangle\}} \langle x|f\rangle = f(x)$

$\|$

$\int dx\, f(x)\,|x\rangle$

例 $|\psi\rangle = \hat{1}\cdot|\psi\rangle = \int dx\,|x\rangle\langle x|\psi\rangle = \int dx\,\psi(x)|x\rangle$

$\langle f|g\rangle = \langle f|\hat{1}|g\rangle = \langle f|\int dx\,|x\rangle\langle x|g\rangle$

$= \int dx\,\langle f|x\rangle\langle x|g\rangle = \int dx\, f^*(x)\,g(x)$

微分算符.

$\langle x|\hat{D}|f\rangle = \frac{d}{dx}f(x) = \frac{d}{dx}\langle x|f\rangle$

$\langle x|\hat{D}|x'\rangle = \frac{d}{dx}\langle x|x'\rangle = \frac{d}{dx}\delta(x-x')$

$\langle f|\hat{D}|g\rangle$
$= \langle f|\hat{1}\hat{D}|g\rangle$
$= \int dx \langle f|x\rangle\langle x|\hat{D}|g\rangle$
$= \int dx\, f^*(x)\frac{d}{dx}g(x)$

$\hat{D}$ 是否厄米

$\langle f|\hat{D}|g\rangle \overset{?}{=} \langle g|\hat{D}|f\rangle^* = \left(\int dx\, g^*(x)\frac{d}{dx}f(x)\right)^*$

$\parallel$

$\int dx\, f^*\frac{d}{dx}g(x)$

$= \cancel{f^*(x)g(x)\Big|} - \int dx \frac{df^*(x)}{dx}g(x)$

$\Rightarrow \langle f|\hat{D}|g\rangle = -\langle g|\hat{D}|f\rangle^* = -\langle f|\hat{D}^\dagger|g\rangle$

反厄米).

可定义 $K=-i\hat{D}$ 为厄米

后来可知 $\hbar\hat{K}$ 是动量算符

设 $\hat{K}|k\rangle = k|k\rangle$, $k\in\mathbb{R}$, $\langle k|k'\rangle=0$

$\langle x|\hat{K}|k\rangle = \langle x|k|k\rangle = k\langle x|k\rangle$

$\parallel$

$-i\langle x|\hat{D}|k\rangle$

$= -i\frac{d}{dx}\langle x|k\rangle = \phi_k(x)$

$\Rightarrow -i\frac{d\phi_k(x)}{dx} = k\phi_k(x)$

$\Rightarrow \phi_k(x) = Ae^{ikx}$, $k\in\mathbb{R}$

连续谱($-\infty<k<+\infty$)

$A=?$ $\Rightarrow \langle k|k\rangle = 1$

$\Downarrow$

$\langle k|k'\rangle = \langle k|\hat{1}|k'\rangle$

$= \int_{-\infty}^{+\infty} dx \langle k|x\rangle\langle x|k'\rangle$

$= A^*A'\boxed{\int_{-\infty}^{+\infty} dx\, e^{i(k'-k)x}} = 2\pi\delta(k-k')$

$= 2\pi A^*A'\delta(k-k') \overset{k\neq k'}{=\!=} 0$

$\underset{}{\overset{k=k'}{=\!=}} 2\pi|A|^2\delta(0) \overset{?}{=} 1$

不可归一化,

只可归 $\sqrt{\delta(0)}$ 化, 取 $A=\frac{1}{\sqrt{2\pi}}$

$= \delta(0)$

$\Rightarrow \langle k|k'\rangle = \delta(k-k')$

$\{|k\rangle\}$ 归 $\sqrt{\delta(0)}$ 正交的本征向量组

$|\psi\rangle = \int_{-\infty}^{+\infty} dk\, \tilde{\psi}(k)|k\rangle$

$\langle k'|$

$\Rightarrow \langle k'|\psi\rangle = \int_{-\infty}^{+\infty}\tilde{\psi}(k)\langle k|k'\rangle = \tilde{\psi}(k')$

$\Rightarrow |\psi\rangle = \int_{-\infty}^{+\infty} dk \langle k|\psi\rangle|k\rangle = \left(\int_{-\infty}^{+\infty} dk|k\rangle\langle k|\right)|\psi\rangle$.

$\Rightarrow \int_{-\infty}^{+\infty} dk|k\rangle\langle k| = \hat{1}$

$\hat{A} = \hat{1}\cdot\hat{A}\cdot\hat{1} = \int_{-\infty}^{+\infty}dx'\int_{-\infty}^{+\infty}|x'\rangle\underbrace{\langle x'|\hat{A}|x\rangle}_{f(x)\delta(x-x')}\langle x|$

$= \int_{-\infty}^{+\infty}dx'\int_{-\infty}^{+\infty}dx\, dx\, A(x,x')|x'\rangle\langle x|$

可以找到，$\hat{X}$ ~~作为~~ 坐标算符

$$\hat{X}|x\rangle = x|x\rangle$$

$$\int \hat{X} = \int_{-\infty}^{+\infty} x\,|x\rangle\langle x|$$

$\int_{-\infty}^{+\infty} dx\, f(x)\,|x\rangle\langle x|$

本征向量 $|x\rangle$

值 $f(x)$

$\hat{k}$ 动量算符

坐标空间表示

$$|\psi\rangle = \hat{1}|\psi\rangle = \int_{-\infty}^{+\infty} dx\,|x\rangle\langle x|\psi\rangle = \int_{-\infty}^{+\infty} dx\,\psi(x)\,|x\rangle$$

动量空间表示

$$|\psi\rangle = \hat{1}|\psi\rangle = \int_{-\infty}^{+\infty} dk\,\psi(k)\,|k\rangle$$

转化

$$\cancel{\psi(x) = \langle x|\hat{1}|\psi\rangle = \int dk\,\langle x|k\rangle\langle k|\psi\rangle}$$

$$\cancel{= \frac{1}{\sqrt{2\pi}}\int dk}$$

量子力学（六个基本公设）

一、量子体系的纯态由 Hilbert 空间中一个向量完全描述

没有隐变量

（贝尔不等式）

二、————的物理量由厄米算符描述

$$\begin{cases} q_l \longrightarrow \hat{X}_l \\ p_l \longrightarrow \hat{P}_l \end{cases} \Rightarrow^{\text{直和}} [\hat{X}_l, \hat{P}_k] = i\hbar\,\delta_{lk}.$$

$$F(q_l, p_l) \longrightarrow \hat{F} = \hat{F}(\hat{X}_l, \hat{P}_l)$$

三、任何物理量的本征向量组形成一组（可归一 / $\sqrt{\delta(\cdot)}$）正交基底

四、测量某个物理量时 ———— 本征值

五、塌缩

六、Schrödinger 方程