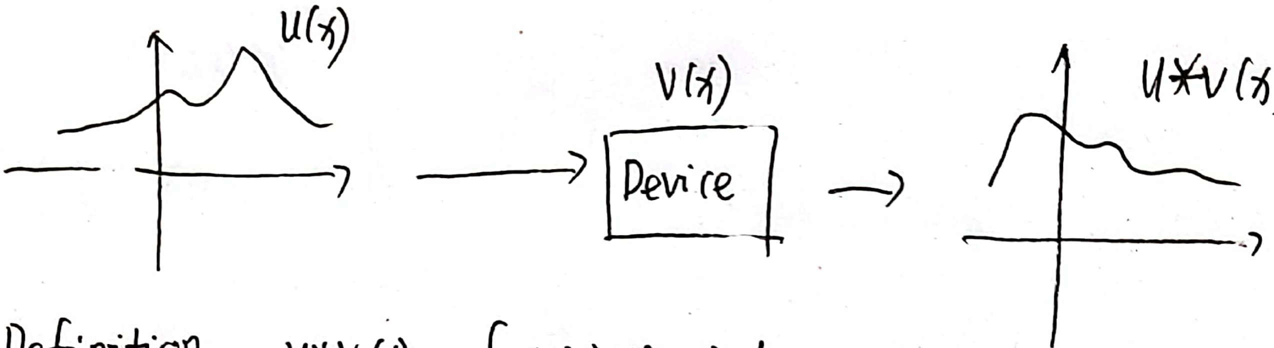
δ Function

Using series to approach it

1.1 convolution: $u * v(x)$



Definition : $u * v(x) = \int_{R} u(y)\, v(x-y)\, dy$ $\quad (u * v = v * u)$

Sufficient condition (i) $|u|^2$ and $|v|^2$ is integrable

(ii) one of $|u|$, $|v|$ is integrable, and another is bounded

(iii) one of u, v has compact support

Differentiation: $D^k(u * v) = u * D^k(v)$

1.2 δ function family

We want a function: $(f * \delta)x = f(x)$ (just like Identity Element)

(There's not such a function in fact, but we can approach it)

δ function family: $\Delta_a : R \to R$ (depends on $a \in A$)

(a) $\Delta_a(x) \geq 0$ (b) $\int_R \Delta_a(x) = 1$ (c) For arbitary neighbourhood of $0 \in R$,

$\lim_{B} \int_U \Delta_a(x) = 1$ (or: $\int_{-\varepsilon}^{\varepsilon} \Delta_a(x) = 1$)

example: $\Delta_n(x) = \begin{cases} \dfrac{(1-x^2)^n}{\int_{|x|<1} (1-x^2)^n dx}, & |x|<1, \\ 0, & |x|>1 \end{cases}$

$\Delta_n(x) = \begin{cases} \dfrac{\cos^{2n} x}{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{2n} x\, dx}, & |x| \leq \frac{1}{2}\pi \\ 0, & |x| > \frac{\pi}{2} \end{cases}$

And Now:

If the function: $R \to C$ is bounded and uniformly continuous, and $\{\Delta_\alpha, \alpha \in A\}$ is δ function family (when $\alpha \to \omega$), Then

$$\alpha \to \omega \Rightarrow (f * \Delta_\alpha)(x) \rightrightarrows f(x)$$

Proof: $|(f * \Delta_\alpha)(x) - f(x)| = |\int_R f(x-y)\Delta_\alpha(y)\,dy - f(x)|$

$$= |\int_R [f(x-y) - f(x)]\Delta_\alpha(y)\,dy| \le \int_{-\delta}^{\delta} |f(x-y) - f(x)|\Delta_\alpha(y)\,dy + \int_{\delta}^{+\infty} |f(x-y) - f(x)|\Delta_\alpha(y)\,dy + \int_{-\infty}^{-\delta} |f(x-y) - f(x)|\Delta_\alpha(y)\,dy$$

$$< \varepsilon \int_{-\delta}^{\delta} \Delta_\alpha(y)\,dy + 4M \int_{\delta}^{+\infty} \Delta_\alpha(y)\,dy$$

$$< \varepsilon + 4M \cdot \frac{\varepsilon}{4M} = 2\varepsilon$$

Corollary: In a closed interval, algebraic polynomial could approach all the ~~consist~~ continuous function consistently. (uniform convergence)

Proof: $\Delta_n(x) = \begin{cases} \dfrac{(1-x^2)^n}{\int_{-1}^{1} (1-x^2)^n dx}, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$, so when it comes to $[-1,1]$,

$$F * \Delta_n(x) = \int_{-\infty}^{+\infty} F(y)\,\Delta_n(x-y)\,dy = \int_{-\infty}^{+\infty} F(y) \cdot \Delta_n(x-y)\,dy$$

$$= \int_{-1}^{1} F(y) \cdot P_n \cdot [1 - (x-y)^2]^n\,dy = \int_{-1}^{1} F(y) \sum_{k=0}^{2n} \cdot a_k(y)\,x^k\,dy$$

$$= \sum_{k=0}^{2k} \cdot b_k \cdot x^k$$

So $\sum_{k=0}^{2k} b_k \cdot x^k \rightrightarrows F(x)$

Similarly, $T_n(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + \sum_{k=1}^{\infty} b_k \sin kx$ is also ok

Proof: $\delta_n(x) = \begin{cases} \dfrac{\cos^{2n} x}{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{2n} x \, dx}, & |x| \leq \frac{\pi}{2} \\ 0, & |x| > \frac{\pi}{2} \end{cases}$, for the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$,

$$F * \delta_n(x) = \int_{-\infty}^{+\infty} F(y)\, \delta_n(x-y)\, dy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(y) \frac{\cos^{2n}(x-y)}{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{2n} x' dx'} \cdot dy$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(y) \sum_{k=0}^{\infty} (a'_k \cos kx + b'_k \sin kx) \cdot dy$$

$$= \sum_{k=0}^{\infty} (a_k \cos kx + b_k \sin kx)$$

How to make the family?

If $\varphi(x) > 0$, $\varphi(x)$ has compact support, and $\int_{-\infty}^{+\infty} \varphi(x)\, dx = 1$,

Then: $\delta_a = \frac{1}{a} \varphi(\frac{x}{a})$ $(a \to 0^+)$ is δ function family

1.3 Distributions / Generalized Functions

1.3.1 Test Function Space

P: linear space generated by functions, and convergence is defined (as pointwise convergence)

P': $\varphi \xrightarrow{u} \begin{cases} k \in C \\ k \in R \end{cases}$ noted as $\langle u | \varphi \rangle = k$ $\quad \begin{cases} \varphi_j \to 0, \ \langle u, \varphi_j \rangle \to 0 \\ \langle u | \lambda_1 \varphi_1 + \lambda_2 \varphi_2 \rangle = \lambda_1 \langle u | \varphi_1 \rangle + \lambda_2 \langle u | \varphi_2 \rangle \end{cases}$

So P is test function space, P' is distribution space

The convergence of P': $A_n \to A := \forall \varphi \in P \ (A_n(\varphi) \to A(\varphi))$ (pointwise convergence)

1.3.2. D and D'

D: linear space of functions that have compact support and could be differentiated for infinite times, and the compact support could be contained by G. We could write it as $C_0^\infty(G, C)$

compact support — the field (of functions)

$C(C_0^\infty(G, C))$ is wider)

The convergence: uniform convergence

① $\forall \alpha$, $j \to \infty \Rightarrow \frac{d^\alpha \varphi_j}{dx^\alpha} \rightrightarrows \frac{d^\alpha \varphi}{dx^\alpha}$ ② all the compact support G_i of φ_i obeys that $G \subseteq K$, where K is compact as well.

D': $\begin{cases} \langle u|\varphi\rangle := \varphi \xrightarrow{u} \begin{cases} k \in R \\ k \in C \end{cases} \\ \langle u|\lambda_1\varphi_1+\lambda_2\varphi_2\rangle = \lambda_1\langle u|\varphi_1\rangle + \lambda_2\langle u|\varphi_2\rangle \\ \text{if } \varphi_j \to 0 \text{ when } j\to\infty, \text{ then } \langle u|\varphi_j\rangle \to 0 \text{ when } \varphi_j \to \infty \end{cases}$, and D' is linear space of u.

and as for the convergence of D':

$\lim_{j\to\infty} u_j \to u := j\to\infty \left(\langle u_j|\varphi\rangle \rightrightarrows \langle u|\varphi\rangle\right)$ (uniform convergence)

example: regular generalized function

$$F(\varphi) = \int_G f(x)\varphi(x)\,dx, \quad \varphi(x) \in D \; \left(\text{or: } F(\varphi) = \int_G f^*(x)\varphi(x)\,dx\right)$$

So all the functions have the distribution related to them.

Just like bra $\langle\psi|$ and ket $|\psi\rangle$, $\langle\psi|$ could be taken as generalized function of $|\psi\rangle$, so $\langle\psi|\varphi\rangle = \int \psi^*(x)\varphi(x)\,dx$ (one dimension)

And δ is not regular generalized function.

δ: $\langle\delta|\varphi\rangle = \varphi(0)$

1.3.3 The differentation of ~~gne~~ generalized function.

By the way, let's learn something about multiplication

Notice that $\int (f\cdot g)(x)\, \varphi(x)\, dx = \int f(x)\, (g\cdot\varphi)(x)\, dx$

That is $\langle f\cdot g | \varphi \rangle = \langle f | g\varphi \rangle$ (regular distribution)

So if we set ~~F∈~~ $F \in D'$, $g \in C^{(\infty)}$,

$$\langle F\cdot g | \varphi \rangle = \langle F | g\varphi \rangle.$$

Practice: prove $\langle \delta\cdot g | \varphi \rangle = g(0)\, \varphi(0)$

Now we return to differentation

Notice that $\int_R f'(x)\, \varphi(x)\, dx = -\int_R f(x)\, \varphi'(x)\, dx$

So $\langle F' | \varphi \rangle = -\langle F | \varphi' \rangle$

Practice: 1. $H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0, \end{cases}$ figure out $H'(x)$ (in terms of distribution)

Solution. $\langle H' | \varphi \rangle = -\langle H | \varphi' \rangle = -\int_{-\infty}^{+\infty} H(x)\, \varphi'(x)\, dx = -\int_0^{+\infty} \varphi'(x)\, dx$

$$= -\varphi(x)\Big|_0^{+\infty} = \varphi(0)$$

($\varphi(x) \in D$, so $\lim_{x\to\infty} \varphi(x) = 0$)

and $\varphi(0) = \langle \delta | \varphi \rangle$. Thus, $H'(x) = \delta(x)$

2. Prove that $\langle \delta^{n} | \varphi \rangle = (-1)^{n} \cdot \varphi^{n}(0)$

1.3.4 convolution and Green 函 function

Notice that $u * v(x) = \int_{R} u(y) v(x-y)\, dy$

So $\langle u * v | \varphi \rangle = \iint u(y) v(x-y) \varphi(x)\, dx\, dy$

Now we set $F \in D'$, $\psi \in D$,

we have
$$\cancel{\langle F * \psi(x) |}$$
$$\cancel{\langle F * \psi |} = \langle \int F(y) \psi(x-y)\, dy | = F * \psi(x)$$
$$\langle F * \psi |$$

That is, $\langle F * \psi | \varphi \rangle = \langle \int F(y) \psi(x-y)\, dy | \varphi(x) \rangle$

$$= \iint F(y) \psi(x-y) \varphi(x)\, dy\, dx$$

$P \to \cancel{\varphi} P \qquad P' \to P'$

Operator: $A \cancel{X}$: $D \to D$ or $D' \to D'$

(function space to function space)

example. $A = \frac{\partial}{\partial x}$, $A = \nabla^2$ ...

And we set green function E : $A(E) = \delta$

Why do we set that? :

$$A(\varphi) = f(x)$$

Now notice that $A \langle \varphi | = \langle f |$,

$$A(E * f) = A(f * E) = f * A(E) = f * \delta = \delta * f$$
$$= \langle \int \delta(y) f(x-y)\, dy | = \langle f |$$

As a result, $E * f$ is the solution $\langle \varphi |$

Thus, $\langle E * f | x \rangle = \langle \varphi | x \rangle$ $\cancel{*}$, and $\varphi(x) = \int E(x-y) f(y)\, dy$

Transformations in general

1. Linear and Multilinear Transformations

Definition: linear: $A(x_1+x_2)=A(x_1)+A(x_2)$

multilinear: $X_1 \times X_2 \cdots \times X_n \to Y$ denoted as $\mathcal{L}(X_1, \cdots X_n, Y)$

linear with respect to each variable

2. The Norm of Transformation:

Definition: let A be a multilinear transformation $X_1 \times X_2 \times \cdots \times X_n \to Y$

$$\|A\| = \sup_{\substack{x_1 \cdots x_n \\ \neq 0}} \frac{|A(x_1, x_2 \cdots x_n)|_Y}{|x_1|_{x_1} |x_2|_{x_2} \cdots |x_n|_{x_n}}$$ (Y is a normed space)

Notice that it's linear, so

$$\|A\| = \sup_{\substack{x_1 \cdots x_n \\ \neq 0}} \left| A\left(\frac{x_1}{|x_1|}, \frac{x_2}{|x_2|}, \cdots, \frac{x_n}{|x_n|}\right)\right| = \sup_{e_1 \cdots e_n} |A(e_1, \cdots e_n)|$$

It's obvious that $|Ax| \le \|A\||x|$

So there is the definition:

A multilinear transformation $A: X_1 \times X_2 \times \cdots \times X_n \to Y$ is bounded if there's $M \in R$ makes $|A(x_1, x_2 \cdots x_n)| \le M|x_1||x_2| \cdots |x_n|$ for all $x_1, x_2 \cdots, x_n$

Notice: operators in quantum mechanics like $\hat{p} = -i\hbar\frac{\partial}{\partial x}$ and $\hat{x} = x$ aren't bounded.

3. Continous Transformations

Proposition: for $A: X_1 \times X_2 \times \cdots \times X_n \to Y$

those are the same,

(a) A has a finite norm

(b) A ~~has~~ is a bounded transformation

(c) A is a continuous transformation. "Continumous" means that

when $|x_i - x_i'| < \delta$, $|A(x_1 \cdots x_i + x_i' \cdots x_n) - A(x_1 \cdots x_i \cdots x_n)| < \varepsilon$

(d) A is continuous at $(0, 0, \cdots 0)$

supplement: $\|B \circ A\| \leq \|B\| \|A\|$

4. The differential of a Mapping

Definition: X and Y is normed spaces. $f: E \to Y$ at the set $E \subset X$ Then f is differential at the point $x \in E$ if there's a continous linear transformation $L(x): X \to Y$ such that

$$f(x+h) - f(x) = L(x) \cdot h + \alpha(x; h) \quad (h \to 0)$$

where $\alpha(x; h) = o(h)$

example: ① $S = \int_{t_1}^{t_2} L(t, q, \dot{q})\, dt$

$$\delta S = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q}\right) dt = \frac{\partial L}{\partial \dot{q}}\delta q \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right] \delta q\, dt = 0$$

$$\therefore \frac{\partial L}{\partial q} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)$$

② $A \in \mathcal{L}(X; X)$ define: $\exp: \mathcal{L}(X; X) \to \mathcal{L}(X; X)$

$$\exp A = E + \frac{1}{1!}A + \frac{1}{2!}A^2 + \cdots + \frac{1}{n!}A^n$$

Notice that

$\exp(A+h) = \exp(A) + L(A)h + o(h)$, where

$$L(A)h = h + \frac{1}{2!}(Ah+hA) + \cdots + \frac{1}{n!}(A^{n-1}h + A^{n-2}hA + \cdots + AhA^{n-2} + hA^{n-1}) + \cdots$$

And $\|L(A)\| \leq \exp\|A\|$, so $L(A) \in \mathcal{L}(\mathcal{L}(\mathcal{L}(X;X), \mathcal{L}(X;X)))$

is differentiable

5. Finite-Increment Theorem

$f: U \to Y$, $U \in X$, X and Y are normed.

The closed interval $[x, x+h] = \{\xi \in X \mid \xi = x + \theta h, 0 \leq \theta \leq 1\}$ is contained in U and f is differentiable at all points of $(x, x+h)$,

then $|f(x+h) - f(x)|_Y \leq \sup\limits_{\xi \in (x, x+h)} \|f'(\xi)\|_{\mathcal{L}(X,Y)} |h|_X$

6. higher-order derivatives

Definition: $f^n(x) = (f^{n-1})'(x)$

$$f^n(x) \in \mathcal{L}(X; Y_n) = \mathcal{L}(X; \mathcal{L}(X; Y_{n-1})) = \cdots$$
$$= \mathcal{L}(X; \mathcal{L}(X; \mathcal{L}(X \cdots \mathcal{L}(X;Y))\cdots))$$

Thus, we have Taylor series:

$$f(x+h) = f(x) + f'(x)h + \cdots + \frac{1}{n!} f^n(x)h^n + o(|h|^n)$$

To prove that, we ~~ta~~ can take use of Mathematical Induction as well as finite-increment theorem.

Now

$$|f(x+h)-(f(x)+f'(x)h+\cdots+\frac{1}{n!}f^{n}(x)\cdot h^{n}|$$

$$\le |h|\cdot \sup_{0<\theta<1} \| f'(x+\theta h)-(f'(x)+f''(x)h+\cdots+\frac{1}{(n-1)!}f^{n-1}(x)\cdot h^{n-1})\|$$

$$=|h|\cdot o(|\theta h|^{n-1})$$

$$=o(|h|^{n})$$

In fact, we can take use of it to study interior extrema

If we have $f'(x)=0$ $f''(x)=0$ ... $f^{(k-1)}(x)=0$ $f^{(k)}(x)\neq 0$

Then: necessary: k is even and $f^{(k)}(x)h^{k}$ can not take on values of opposite signs (semidefinite, 半定的)

sufficient: $f^{(k)}(x)h^{k}$ on the sphere $|h|=1$ is bounded away from 0, and $f^{(k)}(x)h^{k}>\delta>0$ (minimum) or $f^{(k)}(x)h^{k}<\delta<0$ (maximum) (δ is constant)

example: $f(\vec{r})=f(0)+\nabla f(\vec{r})\cdot\vec{r}+(\nabla\nabla f):(\vec{r}\vec{r})+\cdots+(\underbrace{\nabla\nabla\cdots\nabla}_{n} f)\underbrace{\vdots}_{n}(\underbrace{\vec{r}\vec{r}\cdots\vec{r}}_{n})$

Proof: we can take use of Mathematical Induction

$$\underbrace{\nabla\nabla\cdots\nabla}_{n} f(\vec{r}+d\vec{r})-\underbrace{\nabla\nabla\cdots\nabla}_{n} f(\vec{r})=\underbrace{\nabla\nabla\cdots\nabla}_{n+1} f(\vec{r})\cdot\vec{r}+o(d\vec{r})$$

and using Taylor series, we have the formula.