Ordinary Differential Equations

Second-Order Differential Equations

1.1 homogeneous: $\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)\cdot y=0$

1.1.1 singular point: $P(x)$ or $Q(x)$ diverges when $x\to x_0$

regular: $P(x)$ or $Q(x)$ diverges at x_0 but $(x-x_0)P(x)$, $(x-x_0)^2Q(x)$ remain finite.

irregular: $(x-x_0)P(x)$ ~~and~~ or $(x-x_0)^2Q(x)$ still diverge.

Only the equations with regular singular point could have the series solutions I will introduce later.

1.1.2 Frobenius's Method

We first prove that there's two independent solutions:

We now introduce Wronskian determinant:

$$\begin{vmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ \varphi_1' & \varphi_2' & \cdots & \varphi_n' \\ \vdots & \vdots & & \vdots \\ \varphi_1^{(n-1)} & \varphi_2^{(n-1)} & \cdots & \varphi_n^{(n-1)} \end{vmatrix}$$

It's obvious that when it's not 0, $\varphi_1, \varphi_2 \dots$ and φ_n is linearly independent.

If it's 0, then they are linearly dependent.

Now we return to Second-Order Equations.

If we set: $y_1''+P(x)y_1'+Q(x)\cdot y_1=0$

$y_2''+P(x)\cdot y_2'+Q(x)y_2=0$

We can even get Wronskian directly.

Notice that $W = y_1 y_2' - y_2 y_1'$, and $W' = y_1 y_2'' - y_2 y_1''$

and $y_1 y_2'' + P(x) \cdot y_1 y_2' + Q(x) \cdot y_1 y_2 = 0$

$y_2 y_1'' + P(x) \cdot y_2 y_1' + Q(x) \cdot y_2 y_1 = 0$

$\therefore (y_1 y_2'' - y_2 y_1'') + P(x) \cdot (y_1 y_2' - y_2 y_1') = W' + P(x) W = 0$

Thus, $W = W_0 \exp\left(-\int P(x) dx\right) \neq 0$

So there's at least two linearly independent solutions.

Now if there's 3 solutions,

$$W' = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = -y_1' W_{23}' + y_2' W_{13}' - y_3' W_{12}'$$

$$= P(x) (y_1' W_{23} - y_2' W_{13} + y_3' W_{12})$$

$$= P(x) [y_1' (y_2 y_3' - y_3 y_2') - y_2' (y_1 y_3' - y_3 y_1') + y_3' (y_1 y_2' - y_2 y_1')] = 0$$

$\therefore$ There's only two independent solutions.

Then we'll talk about the series solution.

assume that $w(z) = \sum_{k=0}^{\infty} a_k \cdot (z - z_0)^{k+s}$

and expand $p(x) = P_{-1} \cdot \frac{1}{x - x_0} + P_0 + P_1 \cdot (x - x_0) + \cdots$

$$Q(x) = \frac{Q_{-2}}{(x - x_0)^2} + \frac{Q_{-1}}{(x - x_0)} + Q_0 + Q_1 (x - x_0) + \cdots$$

Notice that the coefficient of $(z - z_0)^{s-2}$ have to be 0, And then all the coefficient have to be 0. (recurrence relation)

example. Bessel Equation:

$$x^2y''+xy'+(x^2-m^2)\cdot y=0 \Rightarrow y''+\frac{1}{x}y'+(1-\frac{m^2}{x^2})\cdot y=0$$

So we have $s(s-1)+s-m^2=0$, and $s=m$ or $s=-m$

Now we talk about $s=m$: we assume that

$$y=\sum_{k=0}^{\infty} a_k\cdot x^{k+s}=\sum_{k=0}^{\infty} a_k\cdot x^{k+m}$$

As the coefficient of x^{k+m} have to be 0 (for all possible values of k), we can conclude that

$$(k+m)(k+m-1)\cdot a_k+(k+m)\cdot a_k-m^2a_k+a_{k-2}=0$$

Thus $a_k=-\frac{a_{k-2}}{(k+m)^2-m^2}=-\frac{a_{k-2}}{(k+2m)\cdot k}$

Notice that $(m+1)\cdot m\,a_1+(m+1)a_1-m^2a_1=(2m+1)a_1=0 \therefore a_1=0$

$$\therefore a_{2k}=-\frac{a_{2k-2}}{4\cdot k(k+m)}=\cdots=(-1)^k\frac{a_0\,m!}{2^{2k}\cdot k!\,(k+m)!}$$

(or $(-1)^k\cdot\frac{a_0\,\Gamma(m+1)}{2^{2k}\cdot k!\,\Gamma(k+m+1)}$ if m isn't an integer)

We often set $a_0=\frac{1}{2^m\Gamma(m+1)}$, So $a_{2k}=(-1)^k\cdot\frac{1}{k!\,\Gamma(k+m+1)}\cdot\frac{1}{2^{m+2k}}$

and $y=\sum_{k=0}^{\infty}(-1)^k\frac{1}{k!\,\Gamma(k+m+1)}\cdot\left(\frac{x}{2}\right)^{2k+m}$

Now we can get one solution at least.

In fact, if the equation $s(s-1)+q_{-1}s+p_{-2}=0$ has two different solutions, we could obtain two formulas:

$$y_1=\sum_{k=0}^{\infty} a_k x^{k+s_1},\quad y_2=\sum_{k=0}^{\infty} b_k\cdot x^{k+s_2}$$

If s_1-s_2 isn't an integer, y_1 and y_2 would be linearly independent. But what if s_1-s_2 is an integer? Now we would have some problems. Assume that $s_1-s_2=h\in\mathbb{N}^*$, Now for y_2, we have:

$$b_h\cdot[(s_2+h)(s_2+h-1)+p_{-1}(s_2+h)+q_{-2}]+b_{h-1}[(s_2+h-1)p_0+q_{-1}]+b_{h-2}[(s_2+h-2)\cdot p_1+q_0]$$
$$+\cdots+b_0\cdot[s_2\cdot p_{h-1}+q_{h-2}]=0$$

(As the coefficient of x^{s+h-2} have to be 0)

That is $b_h[s_1(s_1-1)+p_{-1}s_1+q_{-2}]+b_{h-1}[(s_1-1)p_0+q_{-1}]+\cdots+b_0[s_2 p_{h-1}+q_{h-2}]=0$

Notice that $s_1(s_1-1)+p_{-1}s_1+q_{-2}=0$, So it's impossible to continue ($b_{h-1}, b_{h-2}\cdots b_0$ have already been known through recurrence relation)

(It would be contradictory, unless $(s_2+h-1)p_0+q_{-1}=0\cdots\ s_2 p_{h-1}+q_{h-2}=0$, so $b_h\cdot 0+b_{h-1}\cdot 0+\cdots+b_0\cdot 0=0$, it's OK)

To deal with the problem, let's go back to the Wronskian:

$W=W_0\exp(-\int p(x)dx)$, and $W=y_1y_2'-y_2y_1'=y_1^2\frac{d}{dx}\left(\frac{y_2}{y_1}\right)$

we can set W_0 as 0 (the coefficient of y_1, y_2 could be changed, as αy_1, βy_2 are also solutions)

Now we have $y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2} dx$

This is the most general situation.

Notice that $y_1 = \sum a_k \cdot x^{s_1+k}$, and $p(x) = \frac{P_{-1}}{x} + P_0 + P_1 \cdot x + \cdots$

$$\text{So } \int \frac{e^{-\int p(x)dx}}{y_1^2} dx = \int \frac{1}{x^{2s_1+P_{-1}}} \cdot \frac{e^{k_0 + k_1 x \cdots + k_n x^n + \cdots}}{(\sum a_k x^k)^2} dx$$

$$= \int \frac{1}{x^{2s_1+P_{-1}}} (\sum b_k \cdot x^k) \cdot dx$$

and given $s(s-1) + P_{-1}s + q_{-2} = 0$, $\therefore s_1 + s_2 = -P_{-1} + 1$

If we consider the situation when $s_1 - s_2 = n$, we have

$$2s_1 + P_{-1} = n+1$$

$$\therefore y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2} dx = y_1 \int \frac{1}{x^{n+1}} \cdot (\sum b_k x^k) dx = y_1 \cdot \sum_{k=0}^{\infty} c_k \cdot x^{k-n} + A y_1 \cdot \ln x$$

$$= \sum_{k=0}^{\infty} c_k \cdot x^{k+s_1-n} + A y_1 \cdot \ln x = A y_1 \cdot \ln x + \sum_{k=0}^{\infty} c_k' \cdot x^{k+s_2}$$

example: ~~ALL~~ Neumann function.

$$\lim_{\alpha \to \nu} \frac{J_\alpha(x) \cos \alpha\pi - J_{-\alpha}(x)}{\sin \alpha \pi}$$

$$Y_n(x) = \frac{2}{\pi} (\ln \frac{x}{2} + \gamma) J_n(x) - \frac{1}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} (\frac{x}{2})^{2k-n}$$

$$- \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} [\Phi(k) + \Phi(n+k)] (\frac{x}{2})^{2k+n}$$

and ~~其中~~ $\Phi_p = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p}$, $\Phi(0) = 0$

By the way: could we transform the equation $y''+P(x)y'+Q(x)y=0$ into the form of $z''+q(x)\cdot z=0$?

Now we assume that $z\cdot f=y$, and we have

~~$z\ zf''+f$~~ $fz''+z'(2f'+fP)+z(f''+f'P(x)+fQ(x))=0$

If $2f'+fP=0$, we can conclude that $f=\exp[-\frac{1}{2}\int P(t)\,dt]$

and $z''+[Q(x)-\frac{1}{2}P'(x)-\frac{1}{4}P^2(x)]z=0$

~~Ex~~ Example: $\frac{1}{R}\frac{d}{dr}(r^2\frac{dR}{dr})+\{\frac{2\mu r^2}{\hbar^2}[E-V(\vec{r})]\}=L(L+1)$

we can have $R(r)=\frac{u(r)}{r}$

and $\frac{d^2u}{dr^2}+\{\frac{2\mu}{\hbar^2}[E-V(\vec{r})]-\frac{L(L+1)}{r^2}\}u=0$

1.1.2 INHOMOGENEOUS LINEAR ODES $y''+P(x)y'+Q(x)y=F(x)$ *

Now we can introduce variation of Parameters:

We assume that $y(x)=u_1(x)y_1(x)+u_2(x)y_2(x)$ (Particular Solution)

and the solution is $y_0(x)=y_1(x)\cdot k_1+y_2(x)\cdot k_2+y(x)$

Notice that $y(x)$ has more than one form, since you can add $\alpha y_1+\beta y_2$ $(\alpha,\beta\in z)$ to it.

Now $y'=u_1y_1'+u_2y_2'+(y_1u_1'+y_2u_2')$

Notice that if we put $y(x)$ into the formula (*), we can notice that it's redundant. So we can set $y_1u_1'+y_2u_2'=0$

$$y' = u_1 y_1' + u_2 y_2' \ , \ y'' = u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2' y_2'$$

$$\therefore (u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2' y_2') + P(x)(u_1 y_1' + u_2 y_2') + Q(x)(u_1 y_1 + u_2 y_2) = F(x)$$

$$\therefore \begin{cases} u_1' y_1' + u_2' y_2' = F(x) \\ u_1' y_1 + u_2' y_2 = 0 \end{cases} \qquad \begin{cases} u_1'(x) = -\dfrac{y_2(s) F(s)}{W\{y_1(s), y_2(s)\}} = -\dfrac{y_2(s) F(s)}{W\{y_1(s), y_2(s)\}} \\ u_2'(x) = \dfrac{y_1(s) F(s)}{W\{y_1(s), y_2(s)\}} \end{cases}$$

and we have $y(x) = y_2(x) \int^x \dfrac{y_1(s) F(s) ds}{W\{y_1(s), y_2(s)\}} - y_1(x) \int^x \dfrac{y_2(s) F(s) ds}{W\{y_1(s), y_2(s)\}}$

2. Nonlinear Differential Equations (NDE)

2.1 Bernoulli Equations

$$y'(x) = p(x) y(x) + q(x) [y(x)]^n, \ n \neq 0, \ n \neq 1$$

$$\Downarrow$$

$$\frac{y'(x)}{y^n} = \frac{p(x)}{y^{n-1}} + q(x)$$

$$\Downarrow$$

$$\frac{1}{1-n} \frac{d}{dx}(y^{1-n}) = p(x) \cdot y^{1-n} + q(x)$$

2.2 Riccati Equations $y' = p(x) \cdot y^2 + q(x) \cdot y + r(x)$

There's no general way to deal with it, but if there's special solution y_0, we can write general solution $y = y_0 + u$,

and u satisfy Bernoulli equation $u' = pu^2 + (2py_0 + q)u$

2.3 Fixed singularities : singular points that are independent of the initial or boundary conditions

Movable singularities: vary with initial or boundary conditions

example: $y' = y^2$

if $y(0)=1$. $y(x) = \frac{1}{1-x}$ could be the solution

if $y(0)=2$: $y(x) = \frac{2}{1-2x}$ could be the solution