Partial Differential Equations

1. The Categories

Now we mainly focus on second-order linear partial differential equations. We set all the independent variables as $x_1, x_2 \cdots x_n$, and we can write them in the form:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} u_{x_i} u_{x_j} + \sum_{i=1}^{n} b_i u_{x_i} + cu + f = 0$$

and a_{ij}, b_i, c, f is just the function of $x_1, x_2 \cdots x_n$,

If $f \equiv 0$, then it's homogeneous; if not, it's inhomogeneous.

And due to the linearity, we have superposition principle just like fields.

Now we will learn about equations with two variables:

$$a_{11} u_{xx} + 2a_{12} u_{xy} + a_{22} u_{yy} + b_1 u_x + b_2 u_y + cu + f = 0$$

And we try to have the transformation:

$$\begin{cases} x = x(\xi, \eta) \\ y = y(\xi, \eta) \end{cases} \quad \text{or} \quad \begin{cases} \xi = \xi(x, y) \\ \eta = \eta(x, y) \end{cases}, \text{ and } \frac{\partial(\xi, \eta)}{\partial(x, y)} \neq 0$$

Thus, we have
$$\begin{cases} u_x = u_\xi \xi_x + u_\eta \eta_x \\ u_y = u_\xi \xi_y + u_\eta \eta_y \end{cases}$$

$$u_{xx} = u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}$$

$$u_{yy} = u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}$$

$$u_{xy} = u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y + u_\xi \xi_{xy} + u_\eta \eta_{xy}$$

Thus, we have

$$A_{11}u_{\xi\xi} + 2A_{12}u_{\xi\eta} + A_{22}u_{\eta\eta} + B_1 u_\xi + B_2 u_\eta + Cu + F = 0$$

where
$$\begin{cases} A_{11} = a_{11}\xi_x^2 + 2a_{12}\xi_x\xi_y + a_{22}\xi_y^2 \\ A_{12} = a_{11}\xi_x\eta_x + a_{12}(\xi_x\eta_y + \xi_y\eta_x) + a_{22}\xi_y\eta_y \\ A_{22} = a_{11}\eta_x^2 + 2a_{12}\eta_x\eta_y + a_{22}\eta_y^2 \\ B_1 = a_{11}\xi_{xx} + 2a_{12}\xi_{xy} + a_{22}\xi_{yy} + b_1\xi_x + b_2\xi_y \\ B_2 = a_{11}\eta_{xx} + 2a_{12}\eta_{xy} + a_{22}\eta_{yy} + b_1\eta_x + b_2\eta_y \\ C = c \\ F = f \end{cases}$$

Notice that A_{11} and A_{22} is decided by the same equation:

$$a_{11}z_x^2 + 2a_{12}z_x z_y + a_{22}z_y^2 = 0$$

And it could be turned into ODE:

assume that $z(x,y) = \text{const}$, and we have

$\frac{dy}{dx} = -\frac{z_x}{z_y}$, and we can get that

$$a_{11}\left(\frac{dy}{dx}\right)^2 - 2a_{12}\left(\frac{dy}{dx}\right) + a_{22} = 0 \quad (*)$$

That's so called sealar equation, and its integration $\xi(x,y)=0$ or $\eta(x,y)=0$ is called characteristic line.

After solving $(*)$, we can get that

$$\frac{dy}{dx} = \frac{a_{12} \pm \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}}$$

And deviding through the sign of $a_{12}^2 - a_{11}a_{22}$, we have

$$\begin{cases} a_{12}^2 - a_{11}a_{22} > 0 & ~~\text{elliptic}~~ \text{ hyperbolic} \\ a_{12}^2 - a_{11}a_{22} = 0 & \text{parabolic} \\ a_{12}^2 - a_{11}a_{22} < 0 & ~~\text{ep}~~ \text{ elliptic} \end{cases}$$

(Notice that: $A_{12}^2 - A_{11}A_{22} = (a_{12}^2 - a_{11}a_{22})(\xi_x\eta_y - \xi_y\eta_x)^2$)

1.1 hyperbolic

Now we have real characteristic line $\begin{cases} \xi(x,y) = \text{const} \\ \eta(x,y) = \text{const} \end{cases}$

Now we have $u_{\xi\eta} = -\frac{1}{2A_{12}}[B_1u_\xi + B_2u_\eta + Cu + F]$

If we take $\begin{cases} \xi = \alpha + \beta \\ \eta = \alpha - \beta \end{cases}$, we have

$$u_{\alpha\alpha} - u_{\beta\beta} = -\frac{1}{A_{12}}[(B_1+B_2)u_\alpha + (B_1-B_2)u_\beta + 2Cu + 2F]$$

(like wave equation)

(2) parabolic:

Now we only have one set of characteristic lines:

$\xi(x,y)=0$, So $A_{11}=0$

Notice that $\frac{\xi_x}{\xi_y}=-\frac{dy}{dx}=-\frac{a_{12}}{a_{11}}$ and $a_{12}^2=a_{11}a_{22}$, assume that $\frac{\eta_x}{\eta_y}\neq\frac{dy}{dx}$

$\therefore A_{12}=\xi_y\left[a_{11}\left(\frac{\xi_x}{\xi_y}\right)^2\eta_y+a_{12}\left(\frac{\xi_x}{\xi_y}\eta_y+\eta_x\right)+a_{22}\eta_y\right]$

$=\frac{\xi_y\eta_y}{a_{11}}\left(-a_{12}^2+a_{11}a_{22}\right)=0$

$A_{22}=\eta_y^2\left[\sqrt{a_{11}}\left(\frac{\eta_x}{\eta_y}\right)\pm\sqrt{a_{22}}\right]^2\neq0$

Now we have $u_{\eta\eta}=-\frac{1}{A_{22}}\left[B_1u_\xi+B_2u_\eta+Cu+F\right]$
(like heat equation)

(3) Elliptic

Now the characteristic lines are complex, so

ξ and η are complex, and $\xi=\eta^*$

$u_{\xi\eta}=-\frac{1}{2A_{12}}\left[B_1u_\xi+B_2u_\eta+Cu+F\right]$ (Note: ξ and η are complex)

If we turn them into real independent variables,

we can get that $\begin{cases}\xi=\alpha+\beta\\ \eta=\alpha-\beta\end{cases}$ or $\begin{cases}\alpha=\frac{1}{2}(\xi+\beta)\\ \beta=\frac{1}{2i}(\xi-\beta)\end{cases}$

and now we have

$$u_{\alpha\alpha}+u_{\beta\beta}=-\frac{1}{A_{12}}\left[(B_1+B_2)u_\alpha+i(B_2-B_1)u_\beta+2(u+F\right]$$

Laplace Equation is the most typical one.

2. Wave equation: d'Alembert formula

We are mainly talking about the equation.

$$\left(\frac{\partial^2}{\partial t^2}-a^2\frac{\partial^2}{\partial x^2}\right)u=0 \Rightarrow \left(\frac{\partial}{\partial t}+a\frac{\partial}{\partial x}\right)\circ\left(\frac{\partial}{\partial t}-a\frac{\partial}{\partial x}\right)u=0$$

Now we have the transformation:

$$x=a(\xi+\eta),\ t=\xi-\eta \Rightarrow \begin{cases} a\xi=x+at \\ a\eta=x-at \end{cases}$$

$$\therefore \frac{\partial}{\partial t}+a\frac{\partial}{\partial x}=\frac{\partial}{\partial \xi},\quad \frac{\partial}{\partial t}-a\frac{\partial}{\partial x}=\frac{\partial}{\partial \eta}$$

$$\therefore \frac{\partial^2 u}{\partial\xi\partial\eta}=0 \quad \therefore u=f_1(\xi)+f_2(\eta)$$

$$=f_1\left(\frac{x}{a}+t\right)+f_2\left(\frac{x}{a}-t\right)$$

$$=g_1(x+at)+g_2(x-at)$$

Then how could we define g_1 and g_2? That's impossible without initial condition and boundary condition. Now we will talk about the situation where the solution domain is infinite.

But we still have initial conditions.

That is $\begin{cases} u|_{t=0} = \varphi(x) \\ u_t|_{t=0} = \psi(x) \end{cases}$

and we have $\begin{cases} f_1(x) + f_2(x) = \varphi(x) \\ f_1(x) - f_2(x) = \frac{1}{a}\int_{x_0}^{x} \psi(y)\,dy + f_1(x_0) - f_2(x_0) \end{cases}$

$$\therefore u(x,t) = \frac{1}{2}[\varphi(x+at) + \varphi(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} \psi(y)\,dy$$

That's d'Alembert formula.

3. Separation of Variables.

Now we will talk about the equations in the form

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + a^2 \frac{\partial^2 u}{\partial x^2} = 0 \\ u|_{t=0} = \phi(x),\ u_t|_{t=0} = \psi(x) \\ u|_{x=0} = 0,\ u|_{x=L} = 0 \end{cases}$$

(or $u_t|_{x=0} = 0$ ..., but it have to be homogeneous)

Now we have the inspiration: transform the operators into the operators of time and space respectively. That's $Lu(x,t) = 0 \Rightarrow \begin{cases} L_x X = 0 \\ L_t T = 0 \end{cases}$

So it's reasonable for us to assume that

$u = X(x)\,T(t)$, and get the equation

$$\frac{X''(x)}{X(x)} = \frac{1}{a^2}\cdot\frac{T''(t)}{T(t)}$$ Note that $\frac{d}{dx}\left[\frac{1}{a^2}\frac{T''(t)}{T(t)}\right] = 0$

$$\therefore \frac{X''(x)}{X(x)} = \frac{1}{a^2}\frac{T''(t)}{T(t)} = const = -\lambda$$

And Notice the boundary conditions:

$X(0)T(t) = 0$, $X(L)T(t) = 0$

$\therefore X(0) = 0$ and $X(L) = 0$

And we'll solve the equation:

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = 0 \quad X(L) = 0 \end{cases}$$

If $\lambda \leq 0$, then $X = 0$

If $\lambda > 0$, Then $X_n(x) = B_n \cdot \sin\frac{n\pi}{L}x$ while $\lambda = n^2$ $(n \in N^*)$

So we have $u_n(x,t) = (C_n \cos\frac{an\pi}{L}t + D_n \sin\frac{an\pi}{L}t)\sin\frac{n\pi}{L}\cdot x$

and $u(x,t) = \sum_{n=1}^{\infty}(C_n\cos\frac{an\pi}{L}t + D_n\sin\frac{an\pi}{L}t)\sin\frac{n\pi}{L}\cdot x$

Then we will try to find out C_n and D_n.

That's
$$\begin{cases} \varphi(x) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi}{L}x \\ \psi(x) = \sum_{n=1}^{\infty} D_n\cdot\frac{an\pi}{L}\sin\frac{n\pi}{L}x \end{cases}$$

For Fourier Series, we have

$$C_n = \frac{2}{L}\int_0^L \phi(x)\sin\frac{n\pi x}{L}\,dx$$

$$D_n = \frac{2}{n\pi a}\int_0^L \psi(x)\sin\frac{n\pi x}{L}\,dx$$

E↓

Example:

length: infinite

cylinder conductor

Solution: we assume that the solution is $u = R(\rho)\Phi(\varphi)$

Given $\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2} = 0$, we have

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) = -\frac{\Phi''}{\Phi} = \lambda$$

That is: $\begin{cases} \Phi'' + \lambda\Phi = 0 \\ \rho^2 R'' + \rho R' - \lambda R = 0 \end{cases}$

Note that $\Phi(\varphi + 2\pi) = \Phi(\varphi)$, we have

$\lambda = m^2$, $m \in N$ (NOT N^*!)

and $\Phi(\varphi) = \begin{cases} A, m = 0 \\ A\cos m\varphi + B\sin m\varphi, m \neq 0 \end{cases}$

as for $\rho^2 R'' + \rho R' - \lambda R = 0$ (or $\rho^2 R'' + \rho R' - m^2 R = 0$)

we can take use of the transformation: $\rho = e^t$, and we have

$\frac{d^2R}{dt^2} - m^2R = 0$, and the solution could be written as

$$R(\rho) = \begin{cases} Ce^{mt} + De^{-mt} = C\rho^m + D\cdot\frac{1}{\rho^m}, & m \neq 0 \\ C + Dt = C + D\ln\rho, & m \neq 0 \end{cases}$$

so we have

$$u(\rho,\varphi) = C_0 + D_0\ln\rho + \sum_{m=1}^{\infty} \rho^m (A_m\cos m\varphi + B_m\sin m\varphi) + \sum_{m=1}^{\infty} \rho^{-m} (C_m\cos m\varphi + D_m\sin m\varphi)$$

We have ~~two~~ boundary condition: $u|_{\rho=a} = 0$

Thus, $C_0 = -D_0\ln a$, $C_m = -A_m\rho^{2m} = -A_m a^{2m}$, $D_m = -a^{2m}B_m$

and $u(\rho,\varphi) = D_0\ln\frac{\rho}{a} + \sum_{m=1}^{\infty}[A_m(\rho^m - \frac{a^{2m}}{\rho^m})\cos m\varphi + B_m(\rho^m - \frac{a^{2m}}{\rho^m})\sin m\varphi]$

Note that when $\rho\to\infty$,

$u \to -E_0\rho\cos\varphi$, and given Guass' Law, we have

$-\frac{D_0}{\rho}\cdot 2\pi\rho = \frac{q_0}{\varepsilon_0}$, so we have $A_m = 0$, $B_m = 0$ $(m>1)$, $B_1 = 0$,

$A_1 = -E_0$, $D = -\frac{q_0}{2\pi\varepsilon_0}$

That's $u(\rho,\varphi) = \frac{q}{2\pi\varepsilon_0}\ln\frac{a}{\rho} - E_0(\rho - \frac{a^2}{\rho})\cos\varphi$

then ~~we~~ when could we take use of this method?

There are the prerequisites.

(1) PDE should be linear (proposition principle)

(2) PDE should be homogeneous

(3) boundary conditions should be homogeneous

4. Inhomogeneous PDE.

4.1 Eigenfunctions

Now we will talk about the equations. (example)

$$\begin{cases} \frac{\partial^2 v}{\partial t^2} = a^2 \frac{\partial^2 v}{\partial x^2} + f(x,t) & \text{(inhomogeneous)} \\ v|_{x=0} = 0 \quad v|_{x=L} = 0 & \text{(homogeneous)} \\ v|_{t=0} = 0 \quad \frac{\partial v}{\partial t}\Big|_{t=0} = 0 & \text{(taking use of proposition principle)} \end{cases}$$

Now we will take use of the method of eigenfunctions.

the eigenfunctions are the function sets $X(x)$ we get when $f(x,t)=0$, while taking use of boundary conditions. It's complete, and is enabled to fit the boundary conditions. Note: it's $X(x)$, not $T(t)$.

(Think: why?)

And the solution could be written as

$v(x,t) = \sum_{n=1}^{\infty} g_n(t) X_n(x)$ (Take t as parameter, and take use of the completeness) (Here $X_n(x) = \sin\frac{n\pi}{L} \cdot x$)

Then our task is to find out $g_n(t)$

Put $v(x,t)$ into the partial equation, we have

$$\sum_{n=1}^{\infty} \left(g_n''(t) + \left(\frac{na\pi}{L}\right)^2 g_n(t)\right) \sin\frac{n\pi}{L}x = f(x,t)$$

we also expand $f(x,t)$ to the series of $X_n(x)$

that is: $f(x,t) = \sum_{n=1}^{\infty} f_n(t) \cdot \sin\frac{n\pi}{L} \cdot x$

Here $f_n(t) = \frac{2}{L}\int_0^L f(x,t) \sin\frac{n\pi x}{L} dx$.

And we have $g_n''(t) + \left(\frac{na\pi}{L}\right)^2 g_n(t) - f_n(t) = 0$

(*) Given initial conditions, there's $\begin{cases} g_n(0) = 0 \\ g_n'(0) = 0 \end{cases}$

Think: why it's homogeneous? What if it isn't?

After using Laplace transformation, we have

$g_n(t) = \frac{L}{n\pi a} \int_0^t f_n(\tau) \sin\frac{n\pi a}{L}(t-\tau) d\tau$ (Convolution) example: $f(x,t) = \sin\frac{2\pi x}{L}\sin\frac{2a\pi t}{L}$

Solution: $f_n(t) = \frac{2}{L}\int_0^L \sin\frac{2\pi x}{L}\sin\frac{2a\pi t}{L}\sin\frac{n\pi}{L}x\,dx$

$$= \sin\frac{2a\pi t}{L}\ \delta_{2,n}$$

Thus, $v(x,t) = \sum_{n=1}^{\infty} \delta_{2,n}\left[\frac{L}{n\pi a}\int_0^t \sin\frac{2a\pi\tau}{L}\sin\frac{na\pi}{L}(t-\tau)\,d\tau\right]\sin\frac{n\pi}{L}x$

$$= \frac{L}{4a\pi}\left(\frac{L}{2a\pi} - t\cos\frac{2a\pi}{L}t\right)\sin\frac{2\pi}{L}x$$

Then we will be faced with general problems:

$$\begin{cases} \frac{\partial^2 w}{\partial t^2} = a^2\frac{\partial^2 w}{\partial x^2} + f(x,t) \quad ① \\ w|_{x=0} = 0,\ w|_{x=L} = 0 \\ w|_{t=0} = \phi(x),\ \frac{\partial w}{\partial t}\Big|_{t=0} = \psi(x) \quad ② \end{cases}$$

Notice that all are linear, so we can set

$w(x,t) = u(x,t) + v(x,t)$

where $u(x,t)$ could meet ① and $u|_{t=0} = 0$, $\frac{\partial u}{\partial t} = 0$

$v(x,t)$ could meet ② and $\frac{\partial^2 v}{\partial t^2} = a^2\frac{\partial^2 v}{\partial x^2}$

(In fact, you can also combine them together:

~~and~~ and $g_n(0) = C_n = \frac{2}{l}\int_0^l \phi(x)\sin\frac{n\pi}{l}x\,dx$)

4.2 Impulse Theorem Law

Now we'll talk about wave equation or transport equation

The method could only deal with the situation where initial condition is 0. If not, we can seperate it (as before).

Now we have the equations:

$$\begin{cases} u_{tt} - a^2 u_{xx} = f(x,t) \\ u|_{x=0} = 0 \quad u|_{x=l} = 0 \\ u|_{t=0} = 0, \quad u_t|_{t=0} = 0 \end{cases}$$

(Take $f(x,t)$ as force)

$$\begin{cases} v_{tt} - a^2 v_{xx} = f(x,\tau)\delta(t-\tau) \\ v|_{x=0} = 0 \quad v|_{x=l} = 0 \\ v|_{t=0} = 0 \quad v_t|_{t=0} = 0 \end{cases} \xrightarrow{\text{integration } (\delta \text{ function})} \text{the same solution} \Longleftrightarrow \begin{cases} v_{tt} - a^2 v_{xx} = 0 \\ v|_{x=0} = 0 \quad v|_{x=l} = 0 \\ v|_{t=\tau} = 0 \quad v_t|_{t=\tau} = f(x,\tau) \end{cases}$$

$\Leftarrow$ $v = v(x,t;\tau)$

And notice the definition of convolution:

~~f(x,t)=~~ ~~f(x,t)~~ $f * \delta = \int f(x,\tau)\delta(t-\tau)\,d\tau$.

$\therefore u(x,t) = \int_0^t v(x,t;\tau)\,d\tau$

For $u_t - a^2 u_{xx} = f(x,t)$

$u_x|_{x=0} = 0 \quad u_x|_{x=l} = 0$

$u|_{t=0} = 0$

There's $\begin{cases} v_t - a^2 v_{xx} = f(x,\tau)\,\delta(t-\tau) \\ v_x|_{x=0} = 0 \quad v_x|_{x=l} = 0 \\ v|_{t=0} = 0 \end{cases} \Longleftrightarrow \begin{cases} v_t - a^2 v_{xx} = 0 \\ v_x|_{x=0} = 0 \quad v_x|_{x=l} = 0 \\ v|_{t=\tau} = f(x,\tau) \end{cases}$

$\Downarrow$

$v(x,t;\tau)$

and $u(x,t) = \int_0^t v(x,t;\tau)$

Example: $\begin{cases} u_t - a^2 u_{xx} = A\sin\omega t \\ u|_{x=0} = 0 \quad u_x|_{x=l} = 0 \\ u|_{t=0} = 0 \end{cases}$

Solution: $v_t - a^2 v_{xx} = 0$

$v|_{x=0} = 0 \quad v_x|_{x=l} = 0$

$v|_{t=\tau} = A\sin\omega\tau$

and we have $v(x,t;\tau) = \sum_{n=0}^{\infty} C_n \exp\left[-\frac{(n+\frac{1}{2})^2\pi^2 a^2}{l^2}(t-\tau)\right] \sin\frac{(n+\frac{1}{2})\pi}{l}x$

where $C_n = \frac{2}{l}\int_0^l A\sin\omega\tau \sin\frac{(n+\frac{1}{2})\pi}{l}\xi\, d\xi$

$$= \frac{2A\sin\omega\tau}{(n+\frac{1}{2})\pi}$$

Thus, $u(x,t) = \int_0^t v(x,t;\tau)\,d\tau$

$= \cdots$

$$= \frac{2A}{\pi}\sum_{n=0}^{\infty}\frac{1}{n+\frac{1}{2}}\sin\frac{(n+\frac{1}{2})\pi x}{l}\cdot\frac{1}{(n+\frac{1}{2})^4\frac{\pi^4 a^4}{l^4}+\omega^2}\cdot$$

$$\left\{\frac{(n+\frac{1}{2})^2\pi^2 a^2}{l^2}\sin\omega t - \omega\cos\omega t + \omega\exp\left[-\frac{(n+\frac{1}{2})^2\pi^2 a^2 t}{l^2}\right]\right\}$$

5. Inhomogeneous Boundary Condition

Now take wave equation for example:

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0 \\ u|_{x=0} = \mu(t);\ u|_{x=l} = \nu(t) \\ u|_{t=0} = \varphi(x);\ u_t|_{t=0} = \psi(x) \end{cases}$$

Now we assume that $v(x) = \frac{\nu(t) - \mu(t)}{l}x + \mu(t)$

and $u(x,t) = v(x,t) + w(x,t)$

Now we have $w_{tt} - a^2 w_{xx} = \frac{x}{l}[\mu''(t) - \nu''(t)] - \mu''(t)$

$w|_{x=0} = 0,\quad w|_{x=l} = 0$

$w|_{t=0} = \varphi(x) + \frac{1}{l}[\mu(0) - \nu(0)]x - \mu(0)$

$w_t|_{t=0} = \psi(x) + \frac{1}{l}[\mu'(0) - \nu'(0)]x - \mu'(0)$

*: If $u_x|_{x=0} = \mu(t)$, $u_x|_{x=l} = \nu(t)$: Now we have to change to $v(x,t) = A(t)x^2 + B(t)x$