Further Topics in Analysis

1. Orthogonal Polynomials.

Rodrigues Formulas

考虑一个 ODE

$$p(x)y''+q(x)y'+\lambda y=0$$

$$p(x)=\alpha x^2+\beta x+\gamma \qquad q(x)=\mu x+\nu.$$

写出 y 关于一个正交多项式的形式

$$y_n(x)=\sum_{j=0}^{n} g_j x^j \qquad (g_n\neq 0)$$

将其代入，得到

$$n(n-1)\alpha g_n+n\mu g_n+\lambda g_n=0$$

则 y_n 对应的特征值为 $\lambda_n=-n(n-1)\alpha-n\mu.$

if 方程是自共轭的 $p'(x)=q(x)$.

if 方程不是自共轭的 $\rightarrow w(x)$ weight factor.

$$(wp)'=wq \quad\rightarrow\quad w'=w\frac{q-p'}{p}.$$

$$w(x)=p^{-1}\exp\left(\int^x\frac{q(x)}{p(x)}dx\right).$$

$$\frac{d}{dx}[wpy']+\lambda wy=0.$$

得到 Rodrigues formula

$$y_n=\frac{1}{w}\left(\frac{d}{dx}\right)^n[wp^n(x)]$$

Example: $y''-2xy'+\lambda y=0$

$$p=1 \qquad q=-2x$$

$$w = \exp\left(\int^x (-2x)\,dx\right) = e^{-x^2}$$

代入得:

$$y_n(x) = \frac{(-1)^n}{w}\left(\frac{d}{dx}\right)^n [wp^n] = (-1)^n e^{x^2}\left(\frac{d}{dx}\right)^n e^{-x^2}$$

其使用柯西公式 将多重微分 → 清晰的形式.

Schlaefli Integral:

$$y_n(x) = \frac{1}{w}\frac{n!}{2\pi i}\oint_c \frac{w(z)[p(z)]^n}{(z-x)^{n+1}}\,dz.$$

Generating Functions

给出一组函数 $f_n(x)$

如果 $f(x)$ 可写为 $g(x,t)$ 的幂系数，则称为生成函数

$$g(x,t) = \sum_n C_n f_n(x) t^n = \sum \mathrm{Res}$$

$$C_n f_n(x) = \frac{1}{2\pi i}\oint \frac{g(x,t)}{t^{n+1}}\,dt$$

衍生关系

$$\frac{\partial g(x,t)}{\partial t} = \sum_n nC_n f_n(x)\, t^{n-1} = \sum_n (n+1)C_{n+1} f_{n+1}(x)\, t^n.$$

Example: Hermite Polynomials:

$$e^{-t^2+2tx} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}$$

$$\frac{\partial g}{\partial t} = (2x-2t)\,e^{-t^2+2tx} = \sum_{n=0} nH_n(x)\frac{t^{n-1}}{n!}$$

$$\sum 2x\cdot H_n\frac{t^n}{n!} - \sum 2H_n\frac{t^{n+1}}{n!} = \cdots$$

$$\rightarrow\ 2xH_n(x) - 2nH_{n-1}(x) = H_{n+1}(x)$$

2、Bernoulli Numbers

对于 $\frac{x}{e^x-1} \longrightarrow \sum_{k=0}^{\infty}\frac{B_k x^k}{k!}$

↑ ·

级数展开:

$$f(x)=\frac{x}{e^x-1} \qquad \lim = 1$$

$$f'(x)=\frac{e^x x-e^x+1}{(e^x-1)^2} \qquad \lim = -\frac{1}{2}$$

$$f''(x) = \_\_\_\_ \qquad \lim = -\frac{1}{6}$$

$\vdots$

$$B_n = \lim_{x\to 0}\frac{d^n}{dx^n}\left[\frac{x}{e^x-1}\right]$$

$$B_0 = 1 \quad , \quad \sum_{k=0}^{n-1}\binom{n}{k}B_k = 0 .$$

考虑复变函数 $f(z)=\frac{z}{e^z-1}=\sum B_n\frac{z^n}{n!}$

$\boxed{|z|=1}$ $f(z)$ 是在 C 上解析的.

$$\frac{f(z)}{z^{2n+2}}=\frac{B_0}{0!}z^{-2n-2}+\frac{B_1}{1!}z^{-2n-1}+\dots+\frac{B_{2n+1}}{(2n+1)!}z^{-1}+\frac{B_{2n+2}}{(2n+2)!}$$

利用留数定理

$$\oint_C \frac{f(z)}{z^{2n+2}}dz = 2\pi i\frac{B_{2n+1}}{(2n+1)!}$$

$$B_{2n+1}=\frac{(2n+1)!}{2\pi i}\int_0^{2\pi}\frac{e^{i\theta}}{e^{e^{i\theta}}-1}\cdot e^{-2n i\theta-2i\theta}\, ie^{i\theta}d\theta$$

$e^{i\theta}=z$

$$=\frac{(2n+1)!}{2\pi}\int_0^{2\pi}\frac{e^{-2ni}}{e^{e^{i\theta}}-1}d\theta$$

$$=\frac{(2n+1)!}{2\pi}\underbrace{\int_0^{2\pi}-e^{-2n i\theta}d\theta}=\begin{cases}0 & n\geq 1\\ -\frac{1}{2} & n=0\end{cases}$$

Bernoulli number 在奇项为零.

$$g(x)=\frac{x}{e^x-1}=\frac{x\,(e^{\frac{x}{2}}+e^{-\frac{x}{2}})}{2\,(e^{\frac{x}{2}}-e^{-\frac{x}{2}})} \quad \begin{cases}\sinh=\frac{e^x-e^{-x}}{2}\\ \cosh=\frac{e^x+e^{-x}}{2}\end{cases} \Rightarrow g(x)=\frac{x}{2}\coth\frac{x}{2}=\sum\frac{B_{2n}x^{2n}}{(2n)!}$$

$$\coth x = \sum_{n=0}^{\infty} \frac{2B_{2n}(2x)^{2n-1}}{(2n)!}$$

$x \to ix$ 得

$$\cot x = \sum_{n=0}^{\infty} (-1)^n \frac{2B_{2n}(2x)^{2n-1}}{(2n)!}$$

黎曼 - Zeta 函数.

令 k 为实数 $|k|>1$ 则

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}$$

结论: $\zeta(k) = \prod_{p} \left(\frac{1}{1-p^{-k}}\right)$ p 为质数.

Proof:

$$\left(\frac{1}{1-\frac{1}{p^k}}\right) = 1 + \frac{1}{p^k} + \cdots$$

$$\left(\frac{1}{1-\frac{1}{2^k}}\right)\left(\frac{1}{1-\frac{1}{3^k}}\right) \cdots\cdots$$

$$= \left(1+\frac{1}{2^k}+\frac{1}{2^{2k}}+\cdots\right)\left(1+\frac{1}{3^k}+\cdots\right)\left(1+\frac{1}{5^k}+\cdots\right)\cdots\cdots$$

每一项:

$$\frac{1}{p_1^{m_1 k} p_2^{m_2 k} \cdots p_n^{m_n k}}$$ $m_1, \ldots m_n$ 是整数

根据算术基本定理:

每个正整数都有一个唯一的素数幂的因式分解

$$\left(\frac{1}{1-\frac{1}{2^k}}\right)\left(\frac{1}{1-\frac{1}{3^k}}\right) \cdots = 1 + \frac{1}{2^k} + \cdots \quad \square.$$

另有结论:

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{|B_{2k}|(2\pi)^{2k}}{2(2k)!}$$

Proof:

$$\cot x = \frac{1}{x} + \sum_{n=1}^{\infty}\left(\frac{1}{x+n\pi} + \frac{1}{x-n\pi}\right)$$

$\vdots$

$$= \frac{1}{x} + \sum_{n=1}^{\infty} \frac{1}{\pi n}\left(2\sum_{n=0}^{\infty}\left(\frac{x}{n\pi}\right)^{2k+1}\right)$$

$k \to k+1$

$$\cot x = \frac{1}{x} + \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{2x^{2k-1}}{(n\pi)^{2k}}\right)$$

$$= \frac{1}{x} + \sum_{k=1}^{\infty}\frac{2x^{2k-1}}{\pi^{2k}}\sum_{n=1}^{\infty}\frac{1}{n^{2k}} \to \zeta(2k)$$

$$\cot x = \frac{1}{x} + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{2B_{2k}(2x)^{2k-1}}{(2n)!}$$

$$\zeta(2k) = \frac{|B_{2k}|(2\pi)^{2k}}{2(2k)!} \qquad \square$$

Question：任取两个大于二的整数，其互质的概率是多少？

伯努利多项式：

$$B_k(x,y) \qquad \frac{xe^{xy}}{e^x-1} = \sum_{k=0}^{\infty}\frac{B_k(y)x^k}{k!}$$

$$y=0 \to \frac{xe^{xy}}{e^x-1} = \frac{x}{e^x-1}$$

$$B_k(0) = B_k$$

$$y=1 \to \sum_{k=0}^{\infty}B_k(1)\frac{x^k}{k!} = \frac{xe^x}{e^x-1} = \frac{-x}{1-e^{-x}} = \sum_{k=0}^{\infty}(-1)^k B_k\frac{x^k}{k!}$$

$$B_k(1) = (-1)^k B_k$$

$$B_k(y) = \sum_{n=0}^{k}\binom{k}{n}B_n y^{k-n} \qquad \int_0^1 B_k(y)\,dy = 0 \quad (k \geq 1)$$

另外：$B_k'(y) = kB_{k-1}(y)$

Proof: $\frac{d}{dy}\left(\frac{xe^{xy}}{e^x-1}\right) = \frac{d}{dy}\sum_{k=1}^{\infty}\frac{B_k(y)x^k}{k!}$

$$\Rightarrow \frac{x^2e^{xy}}{e^x-1} = \sum_{k=1}^{\infty}\frac{B_k'(y)x^k}{k!} \quad (k+1 \to k)$$

so: $\frac{xe^{xy}}{e^x-1} = \sum_{k=0}^{\infty}\frac{B_{k+1}'(y)x^k}{(k+1)!}$

$$B_{k+1}'(y) = (k+1)B_k(y)$$

结论：

1、 $B_k(y+1)-B_k(y)=ky^{k-1}$

2、 $B_k(1-y)=(-1)^k B_k(y)$

3、 $B_k(\frac{1}{2})=(2^{1-k}-1)B_k$

3、欧拉麦克劳林公式

$a,b\in \mathbb{Z}$，f 是 $[a,b]$ 上的光滑函数，对于所有 $m\geq 1$ $m\in N$ 有

$$\sum_{i=a}^{b-1} f(i)=\int_a^b f(x)dx+\sum_{k=1}^{m}\frac{B_k}{k!}f^{k-1}(x)\Big|_a^b+R_m$$

$$R_m=(-1)^{m+1}\int_a^b \frac{B_m(y-[y])}{m!}$$

Proof：数学归纳法

1：$a=0$，$b=1$，$m=1$

$$f(0)=\int_0^1 f(x)dx+B_1f(x)\Big|_0^1+R_m=\int_0^1 B_1(x)f'(x)dx$$

且：$f(x)=f(0)+\int_0^x f'(t)dt$

$$\int_0^1 f(x)dx$$

$$=f(0)+\int_0^1\int_0^x f'(t)dtdx$$

$$=w+\int_0^1\int_t^1 f'(t)dxdt$$

$$=w+\int_0^1 f'(t)(1-t)dt \qquad (1)$$

$$=w+\int_0^1 f'(t)dt+\int_0^1 f'(t)(-t)dt.$$

$$=f(1)+\int_0^1 f'(t)(-t)dt \qquad (2)$$

(1)+(2) 得：

$$2\int_0^1 f(x)dx = f(0)+f(1) + \int_0^1 f(t)(1-t)dt + \int_0^1 f(t)(1-t)dt$$

整理得

$$f(0) = \int_0^1 f(x)dx - \frac{1}{2}(f(1)-f(0)) - \int_0^1 f'(x)(\frac{1}{2}-x)dx$$

$$= \int_0^1 f(x)dx + B_1 f(x)\Big|_0^1 + R_m \qquad \square$$

此时，保持 $a=0$ 和 $b=1$ 不变，证明对于 $\forall m \geq 1$ 有

$$f(0) = \int_0^1 f(x)dx + \sum_{k=1}^{m} \frac{B_k}{k!} f^{k-1}(x)\Big|_0^1 + R_m$$

$$R_m = (-1)^{m+1}\int_0^1 \frac{B_m(x)}{m!} f^m(x)dx$$

假设以上对所有 $k \leq m$ 都成立

$$B'_{k+1}(x) = (k+1)B_k(x)$$

考虑 $\int_0^1 B_m(x) f^m(x)dx$.

$$\int_0^1 B_m(x) f^k(x)dx = \frac{B_{k+1}(x)}{k+1} f^k(x)\Big|_0^1 - \frac{1}{k+1}\int_0^1 B_{k+1}(x) f^{k+1}(x)dx$$

代入：

$$R_m = \frac{(-1)^{m+1}}{(m+1)!}\left[B_{m+1}(x) f^m(x)\Big|_0^1 - \int_0^1 B_{m+1}(x) f^{m+1}(x)dx\right]$$

如果 $m \to$ 奇数. $(-1)^{m+1} = 1$ 　　则 $B_{m+1}(0) = B_{m+1}(1) = 0$.

$m \to$ 偶数. $(-1)^{m+1} = -1$

因此，我们得到：

$$\frac{(-1)^{m+1}}{(m+1)!} B_{m+1}(x) f^m(x)\Big|_0^1 = \frac{1}{(m+1)!} B_{m+1} f^m(x)\Big|_0^1.$$

$f(0)$ 值变为

$$f(0) = \int_0^1 f(x)dx + \sum_{k=1}^{m} \frac{B_k}{k!} f^{k-1}(x)\Big|_0^1 + R_m \to \text{代入}.$$

$$= \int_0^1 f(x)dx + \sum_{k=1}^{m+1} \frac{B_k}{k!} f^{k-1}(x)\Big|_0^1 + \frac{(-1)^{m+2}}{(m+1)!}\int_0^1 B_{m+1}(x) f^{m+1}(x)dx \qquad \square$$

对每个整数 $a \le i \le b$ 求和.

$$\sum_{a}^{b-1} f(i) = \sum_{a}^{b-1}\left[\int_i^{i+1} f(x)dx + \sum_{k=1}^{m+1}\frac{B_k}{k!}f^{k-1}(x)\Big|_i^{i+1} + \frac{(-)^{m+1}}{(m+1)!}\int_i^{i+1} B_{m+1}(x) f^{m+1}(x)dx\right]$$

$$= \int_a^b f(x)dx + \sum_{k=1}^{m+1}\frac{B_k}{k!}f^{k-1}(x)\Big|_a^b + \frac{(-)^{m+1}}{(m+1)!}\int_a^b B_{m+1}(x) f^{m+1}(x)dx \qquad \Box$$

也可表示为:

$$\sum_{a}^{b} f(n) = \int_a^b f(x)dx + \frac{f(a)+f(b)}{2} + \sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}\left(f^{2k-1}(b) - f^{2k-1}(a)\right) + R_p$$

$$|R_p| \le \frac{2\zeta(p)}{(2\pi)^p}\int_m^n |f^{(p)}(x)|dx$$

4. Dirichlet Series.

形如 $S(s) = \sum_n \frac{a_n}{n^s}$

typical $a_n = 1$ $\quad \zeta = \sum \frac{1}{n^s}$

Eample: 估算 $\zeta(2)$ 的值.

我们有: $S(a) = \sum_{n=1}^{\infty}\frac{1}{n^2+a^2} = \frac{\pi \coth \pi a}{2a} - \frac{1}{2a^2}$

$$\zeta(2) = \lim_{a\to 0} S(a) = \lim_{a\to 0}\left[\frac{\pi}{2a}\left(\frac{1}{\pi a} + \frac{\pi a}{3} + \dots\right) - \frac{1}{2a^2}\right] = \frac{\pi^2}{6}$$

$$\zeta(2) \simeq \frac{\pi^2}{6}$$

$$\zeta(4) = \frac{\pi^4}{90}.$$

5. 无限乘积.

$$P = \prod_{n=1}^{\infty}(1+a_n)$$

换成累加

$$\ln \prod (1+a_n) = \sum_{n=1}^{\infty}\ln(1+a_n)$$

$\begin{cases} 0 \le a_n < 1 \text{ 时.} \\ \text{其与 } \sum_{n=1}^{\infty} a_n \text{ 同敛散} \end{cases}$

由于 $1+a_n \le e^{a_n}$

$p_n \le e^{S_n}$. 让 $n\to\infty$

$\prod^{\infty}(1+a_n) \le \exp(\sum_{n=1}^{\infty} a_n)$.

展开: $p_n = 1+\sum_{i=1}^{\infty} a_i + \sum\sum a_i a_j + \dots \ge S_n$.

$\prod^{\infty}(1+a_n) \ge \sum a_n$.

$\prod(1-a_n) \to$ Complex. 对于 $a_n<\frac{1}{2}$ 有

$(1-a_n) \le (1+a_n)^{-1}$ $(1-a_n) \ge (1+2a_n)^{-1}$,

Example:

$$\sin z = z\prod_{1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right),$$

$$\cos z = z\prod_{1}^{\infty}\left(1-\frac{z^2}{(n-\frac{1}{2})^2\pi^2}\right).$$

$\sum a_n$ 的敛散?

For sin: $\sum_{n=1}^{\infty} a_n = \frac{z^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} \Rightarrow$ 收敛.

For cos: $\sum_{n=1}^{\infty} a_n = \frac{4z^2}{\pi^2}\sum_{n=1}^{\infty}(2n-1)^{2} \sim \frac{z^2}{2}$

6. 渐近级数. QM → WKB

考虑: $I_1(x) = \int_x^{+\infty} e^{-u} f(u)\,du$

$I_2(x) = \int_0^{\infty} e^{-u} f\left(\frac{u}{x}\right)du$.

$E_1(x) = \int_{-\infty}^{x}\frac{e^{u}}{u}du$.

$-E(-x) = \int_x^{\infty}\frac{e^{-u}}{u}du = E_1(x)$

级数展开:

$$E_1(x) = -\gamma - \ln x - \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n \cdot n!}$$

欧拉常数.

参数积分:

$$I(x,p) = \int_x^{+\infty} \frac{e^{-u}}{u^p} du$$

$$= \frac{e^{-x}}{x^p} - \frac{pe^{-x}}{x^{p+1}} + p(p+1)\int_x^{\infty} \frac{e^{-u}}{u^{p+2}} du.$$

$$= e^{-x}\left(\frac{1}{x^p} - \frac{p}{x^{p+1}} + \frac{p(p+1)}{x^{p+2}} + \dots + (-1)^{n-1} \frac{(p+n-2)!}{(p-1)!\, x^{p+n-1}}\right) + (-1)^n \frac{(p+n-1)!}{(p-1)!} \int_x^{\infty} \frac{e^{-u}}{u^{p+n}} du.$$

$$I(x,p) = \int_x^{\infty} \frac{e^{-u}}{u^p} du.$$

余项 $= I(x,p) - S_n(x,p) = R_n(x,p) = (-1)^{n+1} \frac{(p+n)!}{(p-1)!} \int_x^{\infty} \frac{e^{-u}}{u^{p+n+1}} du.$

其绝对值

$$|R_n(x,p)| \le \frac{(p+n)!}{(p-1)!} \int_x^{\infty} \frac{e^{-u}}{u^{p+n+1}} du \qquad u = v + x$$

$$\Rightarrow \int_x^{\infty} \frac{e^{-u}}{u^{p+n+1}} du = e^{-x} \int_0^{\infty} \frac{e^{-u}}{(v+x)^{p+n+1}} dv.$$

$$= \frac{e^{-x}}{x^{p+n+1}} \int_0^{\infty} e^{-v} \left(1+\frac{v}{x}\right)^{-p-n-1} dv$$

对 $x \to \infty$ $\simeq 1$

$$|R_n(x,p)| \simeq \frac{(p+n)!}{(p-1)!} \frac{e^{-x}}{x^{p+n+1}}$$

如 x 足够大, 部分和可以是原函数的一个很好的近似

因此有时称为半收敛级数.

$E_1(x)$:

$$e^x E_1(x) = e^x \int_x^{\infty} \frac{e^{-u}}{u} du \simeq S_n(x) = \frac{1}{x} - \frac{1!}{x^2} + \dots (-1)^n \frac{n!}{x^{n+1}}$$

$$0.1664 \le e^x E_1(x) \Big|_{x=5} \le 0.1741$$

Cosine and Sine Integrals:

$$Ci(u) = -\int_u^\infty \frac{\cos t}{t} dt$$

$$Si(u) = -\int_u^\infty \frac{\sin t}{t} dt$$

$\to e^{it}$

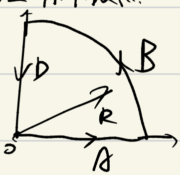
$$Ci(u) + Si(u) = -\int_u^\infty \frac{e^{it}}{t} dt$$

$$F(u) = Ci(u) + iSi(u) = -e^{iu}\int_0^{+\infty} \frac{e^{iz}dz}{u+z}$$

考虑围道积分 $e^{iu}\oint \frac{e^{iz}dz}{u+z}$

当 $u\to\infty$?

在负实轴上有单极点



$R\to\infty$

$D = B + A$

$$D:\quad F(u) = -e^{iu}\int_0^\infty \frac{e^{-y} i\,dy}{u+iy} \qquad \frac{1}{u+iy} = \frac{1}{u}\left[1 - \frac{iy}{u} + \left(\frac{iy}{u}\right)^2 + \dots\right]$$

$$\int_0^\infty y^n e^{-y} dy = n!$$

得到:

$$F(u) \simeq -\frac{ie^{iu}}{u}\left[1 - i\left(\frac{1!}{u}\right) - \left(\frac{2!}{u^2}\right) + i\left(\frac{3!}{u^3}\right) + \dots\right]$$

$$e^{iu} = \cos u + i \sin u$$

$$Ci(u) \simeq \frac{\sin u}{u}\sum_{n=0}^{N}(-1)^n\frac{(2n)!}{u^{2n}} - \frac{\cos u}{u}\sum_{n=0}^{N}(-1)^n\frac{(2n+1)!}{u^{2n+1}}$$

$$Si(u) \simeq -\frac{\cos u}{u}\sum_{n=0}^{N}(-1)^n\frac{(2n)!}{u^{2n}} - \frac{\sin u}{u}\sum_{n=0}^{N}(-1)^n\frac{(2n+1)!}{u^{2n+1}}$$

$$f(x) = \sum^{\infty} a_n$$

$$R_n(x) = f(x) - S_n(x) \qquad S_n(x) = a_0 + \frac{a_1}{x} + \cdots + \frac{a_n}{x^n}$$

$$x^n R_n = x^n [f(x) - S_n(x)]$$

有性质：$\lim_{x\to 0} x^n R_n(x) = 0$ n 恒定

$\lim_{n\to\infty} x^n R_n(x) = \infty$ x 恒定

而 $S_n \to$ 幂级数，$R(x)$ 比 x^{-n-1}

$$f(x) \sim \sum_{n=0}^{\infty} a_n x^{-n}$$

$x \to \infty$，$\sim \to =$