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Chapter 19

Fourier Series.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(s) \cos(ns) ds \quad , n = 0, 1, 2 \cdots$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(s) \sin(ns) ds \quad , n = 1, 2, 3$$

Dirichilet Conditions.

1. Finite discontinuities
2. Finite number of extreme values $\sin \frac{1}{x}$

$\Rightarrow$ Piecewise regular.

Exponential form.

$$f(x) = \frac{1}{2} \cdot e^{i0x} \cdot a_0 + \sum_{n=1}^{\infty} \frac{1}{2} a_n \left(\underline{e^{inx}} + e^{-inx} \right) + \sum_{n=1}^{\infty} \frac{1}{2i} b_n \left(\underline{e^{inx}} - e^{-inx} \right)$$

$$= \frac{1}{2} e^{i0x} \cdot a_0 + \frac{1}{2} \sum (a_n - ib_n) e^{inx} + \frac{1}{2} \sum (a_n + ib_n) e^{-inx}$$

$$= \sum_{n=-\infty}^{+\infty} c_n e^{inx}$$

$c_n = \frac{1}{2}(a_n - ib_n)$. $c_{-n} = \frac{1}{2}(a_n + ib_n)$, $n > 0$

$c_0 = \frac{1}{2} a_0$

Sturm Liouville Theory.

$$\mathcal{L}\psi(x) = \lambda\psi(x)$$

$$\mathcal{L} = \underbrace{p_0(x)}_{=-1}\frac{d^2}{dx^2} + p_1(x)\frac{d}{dx} + p_2(x)$$

If $\mathcal{L}$ is self-adjoint

$$p_0'(x) = p_1(x)$$

$$\mathcal{L} = \frac{d}{dx}\left(p_0\frac{d}{dx}\right) + p_2(x)$$

$$\mathcal{L}u = (p_0u')' + p_2u.$$

Consider.

$$\to \int_a^b v^*\mathcal{L}u\,dx = \int_a^b v^*(p_0u')'\,dx + \int_a^b v^*p_2u\,dx$$

$$= v^*p_0u'\Big|_a^b + \int_a^b\left[-(v^*)'p_0u' + v^*p_2u\right]dx$$

$$= \boxed{\left[v^*p_0u' - (v^*)'p_0u\right]_a^b} + \int_a^b\left[\left[p_0(v^*)'\right]'u + v^*p_2u\right]dx$$

$$\int_a^b(\mathcal{L}v)^*u\,dx \leftarrow$$

$\lambda_u,\ \lambda_v$

$$\underbrace{(\lambda_u-\lambda_v)}_{\neq 0}\underbrace{\boxed{\int v^*u\,dx}}_{=0.} = \underbrace{\left[v^*p_0u' - (v^*)'p_0u\right]_a^b}_{=0}$$

$$-y''(x) = \lambda \cdot y(x)$$
$$y(0) = y(2\pi)$$
$$y'(0) = y'(2\pi)$$

① $\cos nx$, $\sin nx$

② e^{inx} $n \in \mathbb{Z}$

$$\langle \sin nx \mid \sin nx \rangle = \int_{-\pi}^{\pi} \sin^2 x \, dx = \pi .$$
$$\langle e^{inx} \mid e^{inx} \rangle = 2\pi .$$

① $\varphi_n = \frac{\cos nx}{\sqrt{\pi}}$, $\varphi_{-n} = \frac{\sin nx}{\sqrt{\pi}}$ $\varphi_0 = \frac{1}{\sqrt{2\pi}}$

② : $\varphi_n = \frac{e^{inx}}{\sqrt{2\pi}}$

$\mathcal{L}^2$.

Discontinus Functions.

$$|n| \le r$$

$$f_r(x) = \sum_{n=-r}^{r} c_n e^{inx}, \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt$$

So. $$f_r(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \boxed{\sum e^{in(x-t)}} dt$$

$$\sum_{n=-r}^{r} y^n = \frac{y^{-r} - y^{r+1}}{1-y} = \frac{y^{r+\frac{1}{2}} - y^{-(r+\frac{1}{2})}}{y^{\frac{1}{2}} - y^{-\frac{1}{2}}}$$

$$\sum e^{in(x-t)} = \frac{\left(e^{i(x-t)(r+\frac{1}{2})} - e^{i(x-t)(r+\frac{1}{2})}\right)/2i}{\left(e^{i(x-t)\frac{1}{2}} - e^{i(x-t)\cdot(-\frac{1}{2})}\right)/2i}$$

$y \to e^{i(x-t)}$

$$f_r(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\cdot\frac{\sin[(r+\frac{1}{2})(x-t)]}{\sin[\frac{1}{2}(x-t)]}\,dt$$

$$D_r(u) = \frac{\sin[(r+\frac{1}{2})u]}{\sin\frac{1}{2}u}$$ Dirichilet kernel

$$f_r(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\cdot D_r(x-t)\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\,D_r(t)\,dt$$ even

$$= \frac{1}{2\pi}\int_{0}^{\pi}\left[f(x-t)+f(x+t)\right]D_r(t)\,dt.$$

Riemann Lemma.

If $f:(w_1,w_2)\to\mathbb{R}$ . . . .

then,

$$\lambda\to+\infty,\ \lambda\in\mathbb{R} \Rightarrow \int_{w_1}^{w_2} f(x)\,e^{i\lambda x}\,dx\to 0.$$

$$\int_{w_1}^{w_2} f(x)\cos(\lambda x)\,dx\to 0.$$

$$f_r(x) = \frac{1}{2\pi}\int_0^{\delta} \left(f(x-t)+f(x+t)\right)\cdot\frac{\sin(r+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt$$

Dini Condition.

$f:\mathring{U}(x)\to\mathbb{C}$

If $f(x_-)=\lim_{t\to 0^+} f(x-t)$ and $f(x^+)=\lim_{t\to 0^+} f(x+t)$

exist.

Then $$\int_{+0}\frac{\left(f(x-t)-f(x_-)\right)+\left(f(x+t)-f(x_+)\right)}{t}\,dt = \sigma(x)$$

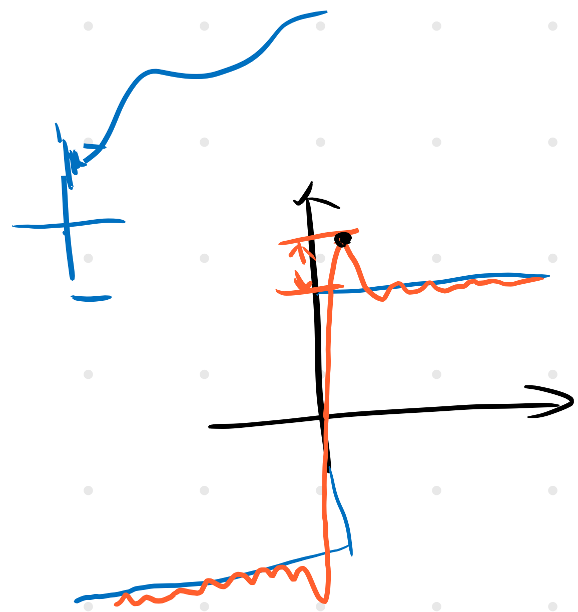
For our case

$$f_r(x) - \frac{f(x_-)+f(x_+)}{2}$$

$$= \frac{1}{\pi}\int_0^{\delta} \frac{(f(x-t)-f(x_-))+(f(x+t)-f(x_+))}{t}\cdot \sin(r+\tfrac{1}{2})t\,dt$$

$$=0.$$

$$f_r(x) = \frac{f(x_+)+f(x_-)}{2}$$

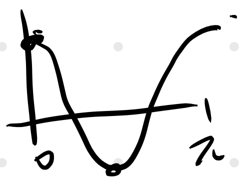


Square wave

$$f(x)=\begin{cases}\frac{h}{2}, & 0<x<\pi\\ -\frac{h}{2}, & -\pi<x<0\end{cases}.$$

$$\frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$=\frac{1}{\sqrt{\pi}}\cdot 2\cdot\int_0^{\pi}\frac{h}{2}\cdot\sin nx\; d(nx)\cdot\frac{1}{n}$$



$$=\frac{h}{n\sqrt{\pi}}\cdot(-\cos nx)\Big|_0^{\pi}.$$

$$=\begin{cases}\frac{2h}{n\sqrt{\pi}}, & \text{odd}\\ 0, & \text{even}\end{cases}$$

So. $$f(x)=\frac{2h}{\pi}\left(\frac{\sin x}{1}+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\cdots\right)$$

$$f_r(x)=\frac{h}{4\pi}\left[\int_0^{\pi}\frac{\sin(r+\frac{1}{2})(x-t)}{\sin\frac{1}{2}(x-t)}dt-\int_{-\pi}^{0}\frac{\sin[(r+\frac{1}{2})(x-t)]}{\sin\frac{1}{2}(x-t)}dt\right.$$

$x-t \to s$ $\qquad$ $x-t \to -s$

$$=\frac{h}{4\pi}\left(\int_{-\pi+x}^{x}-\int_{-\pi-x}^{-x}\right)\frac{\sin(r+\frac{1}{2})s}{\sin\frac{1}{2}s}ds$$

$$\Phi(t) = \int_0^t \frac{\sin(r+\frac{1}{2})s}{\sin\frac{1}{2}s}\,ds$$

$$f_r(x) = \frac{h}{4\pi}\left[\Phi(x) - \Phi(-\pi+x) - \Phi(-x) + \Phi(-\pi-x)\right]$$

$$= \frac{h}{4\pi}\left[\left(\Phi(x) - \Phi(-x)\right) - \left(\Phi(-\pi+x) - \Phi(-\pi-x)\right)\right]$$

So,

$$f_r(x) = \frac{h}{2\pi}\int_0^x \frac{\sin(r+\frac{1}{2})s}{\sin\frac{1}{2}s}\,ds$$

$$\cancel{- \frac{h}{4\pi}\int_{\pi-x}^{-\pi+x} \frac{\sin(r+\frac{1}{2})s}{\sin\frac{1}{2}s}\,ds}$$

$r \to +\infty$, $x \to 0$

$r + \frac{1}{2} \to p$, $ps = \xi$

$$f_r(x) = \frac{h}{4\pi}\int_0^{px} \frac{\sin\xi}{\sin\left(\frac{\xi}{2p}\right)} \cdot \frac{d\xi}{p}.$$

Calculation of Overshoot.

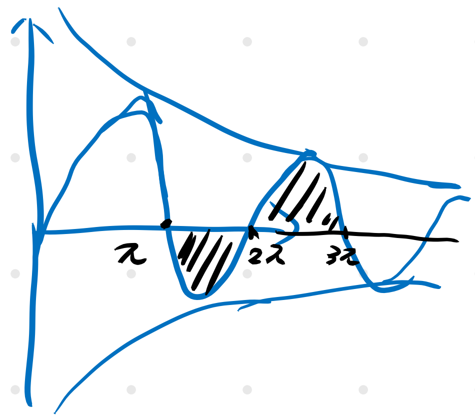
$f_r(0) = 0$.

when $px = \pi$, $f_r(x)$ reaches maximum.

$$f_r(x_{max}) = \frac{h}{\pi}\int_0^{\pi} \frac{\sin\xi}{\xi}\,d\xi = \frac{h}{2} \cdot \boxed{\frac{2}{\pi}\int_0^{\pi} \frac{\sin\xi}{\xi}\,d\xi}$$

$$\int_0^{\pi} \frac{\sin\xi}{\xi}\,d\xi = \int_0^{+\infty} \frac{\sin\xi}{\xi}\,d\xi - \int_{\pi}^{+\infty} \frac{\sin\xi}{\xi}\,d\xi$$

$$= \frac{\pi}{2} - \left(\int_{\pi}^{3\pi} + \int_{3\pi}^{5\pi} + \cdots\right)\frac{\sin\xi}{\xi}\,d\xi$$



$$f_r(x_{max}) = \underline{\left[\frac{2}{\pi}\int_0^{\pi} \frac{\sin\xi}{\xi}\,d\xi\right]} \cdot f(x_{max})$$

$= 1.1789797 \ldots$