Chapter 20.

Integral transforms.

Introduction.

$$g(x) = \int_a^b f(t) K(x,t)\, dt.$$

$$g(x) = \mathcal{L} f(t).$$

Linearity:

$$\int_a^b [f_1(t) + f_2(t)] K(x,t)\, dt$$

$$= \int_a^b f_1(t) K(x,t) dt + \int_a^b f_2(t) K(x,t) dt.$$

$$\int_a^b c f(t) K(x,t)\, dt = c \int_a^b f(t) K(x,t)$$

$$\mathcal{L}^{-1} g(x) = f(t).$$

$$\begin{tikzcd}
\text{Original problem} \arrow[d] \arrow[r, dashed] & \text{Solution} \\
\text{Problem in transform space} \arrow[r] & \text{Solution in transform space} \arrow[u]
\end{tikzcd}$$

Fourier transform.

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) \cdot e^{i\omega t} dt.$$

$t \to \omega$.

Laplace transform

$$F(s) = \int_{0}^{\infty} e^{-ts} f(t) dt$$

differential equations $\to$ algebraic equation

Hankel transform.

$$g(\alpha) = \int_{0}^{+\infty} f(t) \cdot t J_n(\alpha t) dt.$$

Mellin transform

$$g(\alpha) = \int_{0}^{+\infty} f(t) t^{\alpha - 1} dt.$$

example $f(t) = e^{-t}$, $g(\alpha) = \Gamma(\alpha)$.

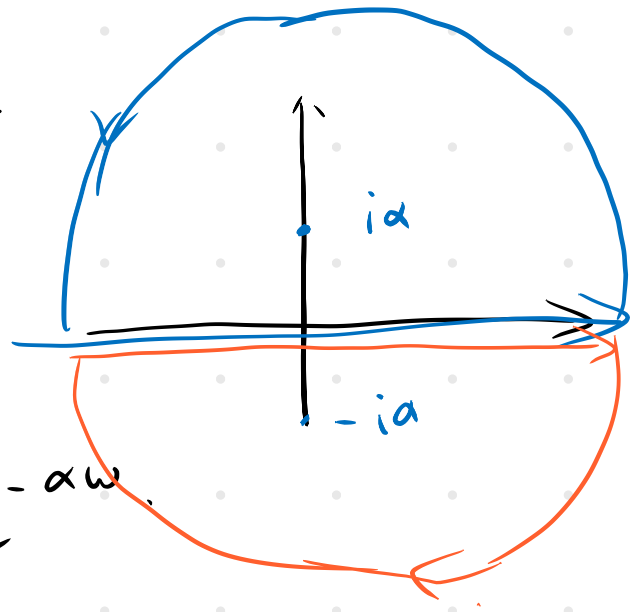
Fourier Transform

example 1. $f(t) = \delta(t)$.

$$g(\omega) = \sqrt{\frac{1}{2\pi}} \cdot \int_{-\infty}^{+\infty} \delta(t) \cdot e^{i\omega t} dt = \sqrt{\frac{1}{2\pi}}$$

example 2. $f(t) = \frac{2\alpha}{\alpha^2+t^2} \cdot \sqrt{\frac{1}{2\pi}}$

$$g(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{2\alpha \cdot e^{i\omega t}}{(t-i\alpha)(t+i\alpha)} \, dt.$$



$t = i\alpha$

$t = -i\alpha$,

① $\omega > 0$

$$g(\omega) = \frac{1}{2\pi} \cdot 2\pi i \cdot \frac{e^{-\alpha\omega}}{i} = e^{-\alpha\omega}.$$

② $\omega < 0$.

$$g(\omega) = \frac{1}{2\pi} (-2\pi i) \cdot \frac{e^{+\alpha\omega}}{-i} = e^{\alpha\omega} = e^{-\alpha|\omega|}.$$

③ $\omega = 0$.

$$g(\omega) = \frac{1}{2\pi} \cdot 2\pi i \cdot \frac{1}{i} = 1.$$

$$g(\omega) = e^{-\alpha|\omega|}$$

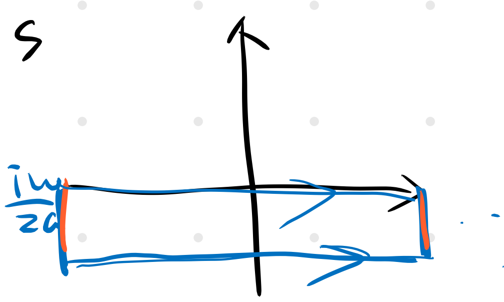
example 3. $f(t) = e^{-at^2}$, $a > 0$.

$$g(\omega) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{+\infty} e^{-at^2+i\omega t} \, dt$$

$$-at^2+i\omega t = -a\left(t^2 - \frac{i\omega t}{a}\right) = -a\left[\underbrace{\left(t-\frac{i\omega}{2a}\right)}_{s}^2 - \frac{\omega^2}{4a^2}\right]$$

$$= \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{+\infty} e^{-a\left[\left(t-\frac{i\omega}{2a}\right)^2 - \underline{\frac{\omega^2}{4a^2}}\right]} d\left(t-\frac{i\omega}{2a}\right).$$

$$= \sqrt{\frac{1}{2\pi}} \cdot e^{-\frac{\omega^2}{4a}} \int_{-\infty-\frac{i\omega}{2a}}^{+\infty-\frac{i\omega}{2a}} e^{-as^2} \, ds$$



$$= \sqrt{\frac{1}{2\pi}} \cdot e^{-\frac{\omega^2}{4a}} \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} e^{-(\sqrt{a}s)^2} \cdot d(\sqrt{a}s)$$

$$= \sqrt{\frac{1}{2\pi}} \, e^{-\frac{\omega^2}{4a}} \cdot \sqrt{\frac{\pi}{a}} = \sqrt{\frac{1}{2a}} \, e^{-\frac{\omega^2}{4a}}$$

Fourier Integral.

$$f(x) = \lim_{n\to+\infty} \int_{-\infty}^{+\infty} f(t)\,\delta_n(t-x)\,dx$$

$$= \lim_{n\to+\infty} \int_{-\infty}^{+\infty} f(t)\cdot\frac{1}{2\pi}\left[\int_{-n}^{n} e^{i\omega(t-x)}\,d\omega\right] dt.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} dt\; f(t)\,\underline{e^{i\omega(t-x)}}$$

$$= \sqrt{\frac{1}{2\pi}}\int_{-\infty}^{+\infty} e^{-i\omega x}\,d\omega \boxed{\int_{-\infty}^{+\infty}\sqrt{\frac{1}{2\pi}}\,f(t)\,e^{i\omega t}\,dt}$$

$$= \sqrt{\frac{1}{2\pi}}\int_{-\infty}^{+\infty} \boxed{e^{-i\omega x}}\,g(\omega)\,dt.$$

$$g_c(\omega) = \sqrt{\frac{2}{\pi}}\int_0^{+\infty} f(t)\cos\omega t\,dt.$$

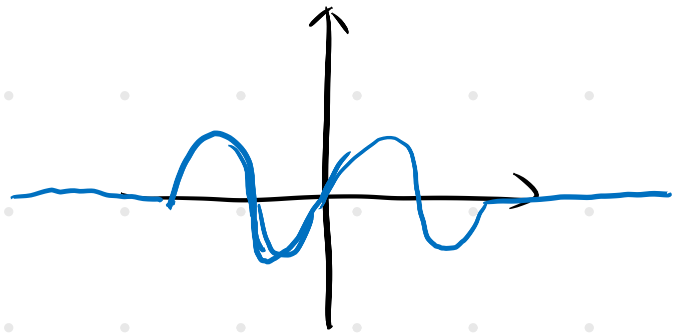
$$f_c(t) = \sqrt{\frac{2}{\pi}}\int_0^{+\infty} g(\omega)\cos\omega t\,d\omega$$

$$g_s(\omega) = \sqrt{\frac{2}{\pi}}\int_0^{+\infty} f(t)\sin\omega t\,dt.$$

$$f_s(\omega) = \sqrt{\frac{2}{\pi}}\int_0^{+\infty} g(\omega)\sin\omega t\,d\omega.$$

example. Finite Wave Train.

$$f(t) = \begin{cases} \sin(\omega_0 t), & |t| \le \frac{N\pi}{\omega_0} \\ 0, & |t| > \frac{N\pi}{\omega_0} \end{cases}$$



$f(t)$ is odd.

$$g_s(\omega) = \sqrt{\frac{2}{\pi}}\int_0^{\frac{N\pi}{\omega_0}} \sin\omega_0 t\,\sin\omega t\,dt.$$

$$\sin\alpha\,\sin\beta = -\frac{1}{2}\left[\cos(\alpha+\beta) - \cos(\alpha-\beta)\right]$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\frac{N\pi}{\omega_0}} -\frac{1}{2}\cdot\left[\cos(\omega_0+\omega)t - \cos(\omega_0-\omega)t\right]dt.$$

$$= \sqrt{\frac{2}{\pi}}\;-\left(\frac{\sin(\omega_0+\omega)t}{2(\omega_0+\omega)} - \frac{\sin(\omega_0-\omega)t}{2(\omega_0-\omega)}\right)\Bigg|_0^{\frac{N\pi}{\omega_0}}$$

$$\sqrt{\frac{2}{\pi}}\; -\left(\frac{\sin(\omega_0+\omega)t}{2(\omega_0+\omega)} - \frac{\sin(\omega_0-\omega)t}{2(\omega_0-\omega)}\right)\Bigg|_0^{\frac{N\pi}{\omega_0}}$$

$$= \sqrt{\frac{2}{\pi}}\cdot\left(\frac{\sin(\omega_0-\omega)\cdot(N\pi/\omega_0)}{2(\omega_0-\omega)} - \cancel{\boxed{\frac{\sin(\omega_0+\omega)(N\pi/\omega_0)}{2(\omega_0+\omega)}}}\right)$$

ω_0 is large. $\omega \approx \omega_0$.

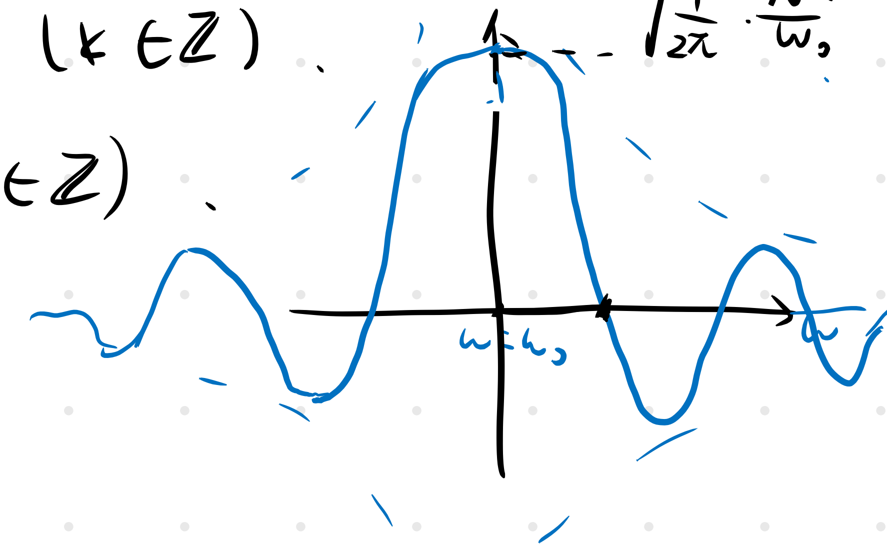
$$\approx \sqrt{\frac{2}{\pi}}\cdot\frac{\sin[(\omega_0-\omega)(N\pi/\omega_0)]}{2(\omega_0-\omega)}$$

Zeros:

$$\Delta\omega\,\frac{N\pi}{\omega_0} = k\pi \quad (k\in\mathbb{Z})$$

$$\frac{\Delta\omega}{\omega_0} = \frac{k}{N}, \quad (k\in\mathbb{Z})$$

$$\boxed{\Delta\omega = \frac{\omega_0}{N}}$$



Transform in 3D-space.

$$g(\vec{k}) = \frac{1}{(2\pi)^{3/2}}\int f(\vec{r})\, e^{-i\vec{k}\cdot\vec{r}}\, d^3r$$

$$f(\vec{r}) = \frac{1}{(2\pi)^{3/2}}\int g(\vec{k})\, e^{-i\vec{k}\cdot\vec{r}}\, d^3k$$

example. Yukawa potential $\frac{e^{-\alpha r}}{r}$.
汤川

$$\boxed{e^{i\vec{k}\cdot\vec{r}}}$$

$$\left[\frac{e^{-\alpha r}}{r}\right]^T(k) = \frac{1}{(2\pi)^{3/2}}\cdot\int \frac{e^{-\alpha r}}{r}\cdot e^{i\vec{k}\cdot\vec{r}}\, d^3r.$$

$$= \frac{4\pi}{(2\pi)^{3/2}}\int_0^{+\infty} r\,dr\int d\Omega_r \sum_{lm} i^l e^{-\alpha r} j_l(kr)\, Y_l^m(\Omega_k)^*\, Y_l^m(\Omega_r).$$

$$Y_0^0 = \frac{1}{\sqrt{4\pi}}.$$

$$\left[\frac{e^{-\alpha r}}{r}\right]^T(\vec{k}) = \frac{4\pi}{(2\pi)^{3/2}} \cdot \int_0^{+\infty} r\, e^{-\alpha r} j_0(kr)\, dr.$$

$$j_0(kr) = \frac{\sin kr}{r}.$$

$$\Rightarrow = \frac{4\pi}{(2\pi)^{3/2}} \int_0^{+\infty} \cdot\, e^{-\alpha r} \cdot \sin kr\, dr.$$

$$= \frac{\quad}{\quad}\ \frac{1}{2}\int_0^{+\infty} e^{-\alpha r}\left(e^{ikr} - e^{-ikr}\right) dr$$

$$\frac{\quad}{\quad}\ \int e^{(ik-\alpha)r} - e^{(-ik-\alpha)r}\, dr$$

$$\frac{\quad}{\quad}\ \frac{1}{2}\left(\frac{-1}{ik-\alpha} + \frac{1}{-ik-\alpha}\right)$$

$$= \frac{1}{(2\pi)^{3/2}} \cdot \boxed{\frac{4\pi}{k^2+\alpha^2}}$$

Coulomb potential $\frac{1}{r}$.

$$\Rightarrow \left[\frac{1}{r}\right]^T(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \cdot \frac{4\pi}{k^2}.$$

Hydrogenic 1s orbital. e^{-Zr}.

$$\left[e^{-Zr}\right]^T(\vec{k}) = -\frac{\partial}{\partial Z}\left[\frac{e^{-Zr}}{r}\right]^T(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \cdot \frac{8\pi Z}{(k^2+Z^2)^2}.$$

Laplace Transforms

Definition

$$f(s) = \mathcal{L}\{F(t)\} = \int_0^{+\infty} e^{-st} F(t)\,dt.$$

$\int_0^{+\infty} F(t)\,dt$. need not exist.

$\mathcal{L}\{e^{t^2}\}$ ✗.

$$\mathcal{L}\{t^n\} = \begin{cases} \checkmark & n > -1 \\ \times & n \leq -1 \end{cases} \quad \text{diverge.}$$

linearity

$$\mathcal{L}\{aF(t) + bG(t)\} = a\mathcal{L}\{F(t)\} + b\mathcal{L}\{G(t)\}.$$

example 1.

$$F(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$\mathcal{L}\{1\} = \frac{1}{s}\int_0^{+\infty} e^{-st}\,d(st) = \frac{1}{s}, \quad s > 0$$

$$F(t) = e^{kt}, \quad t > 0$$

$$\mathcal{L}\{e^{kt}\} = \int_0^{\infty} e^{-st} e^{kt}\,dt = \frac{1}{s-k}, \quad s > k.$$

$$\mathcal{L}\{\cosh(kt)\} = \mathcal{L}\left\{\frac{e^{kt} + e^{-kt}}{2}\right\} = \frac{1}{2}\left(\frac{1}{s-k} + \frac{1}{s+k}\right) = \frac{s}{s^2 - k^2}$$

$$\mathcal{L}\{\sinh(kt)\} = \frac{1}{2}\left(\frac{1}{s-k} - \frac{1}{s+k}\right) = \frac{k}{s^2 - k^2}$$

$$\cos(kt) = \cosh(ikt)$$

$$k \to ik$$

$$\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}$$

$$\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}$$

$s \to 0$. $\mathcal{L}\{\sin kt\} = \frac{1}{k}$.

$$= \int_0^{+\infty} \sin kt \, dt. =$$

$$F(t) = t^n$$

$$\mathcal{L}\{t^n\} = \frac{\int_0^{+\infty} e^{-st}(st)^n \, d(st)}{s^{n+1}} = \frac{\Gamma(n+1)}{s^{n+1}}, \quad s>0, \; n>-1$$

$$F(t) = \delta(t-t_0).$$

$$\mathcal{L}\{\delta(t-t_0)\} = \int_0^{+\infty} e^{-st}\delta(t-t_0)\,dt = e^{-st_0}$$

$$\mathcal{L}\{\delta(t)\} = 1.$$

impulse function.

Inverse Transform.

$$\mathcal{L}^{-1}\{f(s)\} = F(s)$$

$$0 = \mathcal{L}\{F_1(t)-F_2(t)\} = \int e^{-st}[F_1(t)-F_2(t)]\,dt \overset{\Leftarrow}{\not\Rightarrow} F_1(t) = F_2(t).$$

if $\int_0^{t_0} (F_1(t) - F_2(t))\,dt \equiv 0$. (null function).

$$\mathcal{L}\{F_1(t)\} = \mathcal{L}\{F_2(t)\}$$

Lerch's theorem.

differs at isolated points

1. Table

2. Calculus of residues

example$_1$. $f(s) = \dfrac{k^2}{s(s^2+k^2)} = \dfrac{1}{s} - \dfrac{s}{s^2+k^2}$

$\mathcal{L}^{-1}\{f(s)\} = 1 - \cos(kt)$, $t>0$.

example$_2$.

$$F(t) = \int_0^{+\infty} \frac{\sin tx}{x}\,dx$$

$$\mathcal{L}\{k\} = \frac{k}{s}$$

$$\Downarrow$$

$$\mathcal{L}\{\tfrac{\pi}{2}\} = \frac{\pi}{2s}$$

$$\begin{aligned} f(s) &= \mathcal{L}\left\{ \int_0^{+\infty} \frac{\sin tx}{x}\,dx \right\} \\ &= \int_0^{+\infty} e^{-st} \int_0^{+\infty} \frac{\sin tx}{x}\,dx \\ &= \int_0^{+\infty} \frac{1}{x}\left[\int_0^{+\infty} e^{-st} \sin tx\,dt \right] \\ &= \int_0^{+\infty} \frac{dx}{s^2+x^2} \\ &= \frac{1}{s}\arctan\left(\frac{x}{s}\right)\Big|_0^{\infty} \\ &= \frac{\pi}{2s}. \end{aligned}$$

So, $F(t) = \frac{\pi}{2}$, $t>0$.

Properties of Laplace transforms

$$\mathcal{L}\{F'(t)\} = \int_0^{+\infty} e^{-st} \frac{dF}{dt} dt$$

$$= e^{-st} F(t) \Big|_0^{+\infty} + s \boxed{\int_0^{\infty} e^{-st} F(t)\, dt} .$$

$$= s\mathcal{L}\{F(t)\} - F(0^+)$$

$$\mathcal{L}\{F''(t)\} = s\Big[\mathcal{L}\{F'(t)\}\Big] - F'(0^+)$$

$$= s\Big[s\mathcal{L}\{F(t)\} - F(0^+)\Big] - F'(0^+)$$

$$= s^2\mathcal{L}\{F(t)\} - sF(0^+) - F'(0^+)$$

$$\mathcal{L}\{F^{(n)}t\} = s^n\mathcal{L}\{F(t) - s^{n-1}F(0^+) - \cdots\cdots - F^{(n-1)}(0^+)$$

example.

$$\frac{d^2}{dt^2}\sin kt = -k^2 \sin kt$$

$$\mathcal{L}\{\underline{-k^2}\sin kt\} = \mathcal{L}\left\{\frac{d^2}{dt^2}\sin kt\right\}$$

$$= s^2\mathcal{L}\{\sin kt\} - \frac{d}{dt}\sin kt\Big|_{t=0} .$$

$$\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2} .$$

$$f'(s) = \int_0^{+\infty} (-t)\, e^{-st} F(t)\, dt = \mathcal{L}\{-tF(t)\} .$$

$$f^n(s) = \mathcal{L}\{(-t)^n F(t)\} .$$

example: $\mathcal{L}\{e^{kt}\} = \int_0^{\infty} e^{kt-st}\, dt = \dfrac{1}{s-k}$.

$$\frac{1}{(s-k)^2} = \mathcal{L}\{te^{kt}\}$$

Integration of Transforms.

$$f(x) = \int_0^{+\infty} e^{-xt} F(t)\, dt.$$

$$\int_s^{+\infty} f(x)\, dx = \int_s^{+\infty} dx \int_0^{+\infty} dt\, e^{-xt} F(t) = \int_0^{+\infty} e^{-st} \frac{F(t)}{t}\, dt.$$

$$= \mathcal{L}\left\{\frac{F(t)}{t}\right\}.$$

Inverse Laplace Transform

$$F(t) = \mathcal{L}^{-1}\{f(s)\}.$$

extract $e^{\beta t}$

$$F(t) = e^{\beta t} G(t)$$

$$G(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{iut}\, du \int_0^{+\infty} G(v) e^{-iuv}\, dv$$

Insert, we have.

$$F(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{(\beta+iu)t}\, du \int_0^{+\infty} F(v)\, e^{-(\beta+iu)v}\, dv$$

let $s = \beta + iu$ $\quad ds = i\,du$

$$\Rightarrow F(t) = \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} e^{st}\, ds \boxed{\int_0^{+\infty} F(v) e^{-sv}\, dv}$$

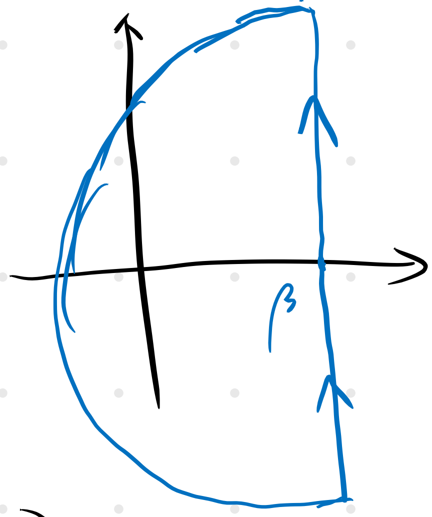
$$F(t) = \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} e^{st} f(s)\, ds.$$

Inverse transform.

Bromwich integral

Fourier-Mellin integral.

$$F(t) = \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} e^{st} f(s)\,ds.$$



$$= \frac{1}{\cancel{2\pi i}} \cdot \cancel{2\pi i}\ \Sigma\left(\text{Residues of } Re(s) < \beta\right).$$

$$F(t) = \Sigma\left(\text{Residues of } Re(s) < \beta\right).$$