Chapter 21

Integral Equations.

Introduction

linear

$$\left.\begin{aligned} f(x) &= \int_a^b K(x,t)\,\varphi(t)\,dt \\ \varphi(x) &= f(x) + \lambda \int_a^b K(x,t)\,\varphi(t)\,dt \end{aligned}\right\} \text{ Fredholm equation}$$

$$\left.\begin{aligned} f(x) &= \int_a^x K(x,t)\,\varphi(t)\,dt \\ \varphi(x) &= f(x) + \int_a^x K(x,t)\,\varphi(t)\,dt \end{aligned}\right\} \text{ Volterra equation}$$

Transformation:

Differential $\longrightarrow$ Integral

linear second-order ODE.

$$y'' + A(x)\,y' + B(x)\,y = g(x).$$

$0 \qquad \omega^2 \qquad 0.$

initial conditions.

$$y(a) = y_0 \ , \quad y'(a) = y_0'$$

Integrate

$$y'(x) = \underline{-\int_a^x A(t)\,y'(t)\,dt} - \int_a^x B(t)\,y(t)\,dt + \int_a^x g(t)\,dt$$

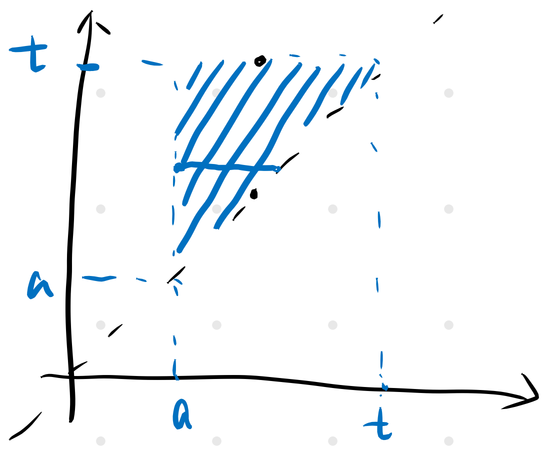
$$= -A(x)\,y(x) + \int_a^x y(t)\,A'(t)\,dt \quad + y_0'$$

$$+ \ \ldots$$

$$= -A(x)\,y(x) - \int_a^x y(t)\left[B(t) - A'(t)\right] dt + \ldots$$

$$y(x) = -\int_a^x A(t)y(t)\,dt - \int_a^x du\int_a^u [B(t) - A'(t)]\,y(t)\,dt + \int_a^x du\int_a^u g(t)\,dt + (A(a)y_0 + y_0')(x-a) + y_0 .$$

To make it neater



$$\int_a^x du\int_a^u f(t)\,dt \rightarrow \int_a^t dy\int_a^y f(x)\,dx$$

$$\downarrow$$

$$\int_a^x f(t)\,dt\int_t^x du \leftarrow \int_a^t dx\int_x^t f(x)\,dy$$

$$\parallel$$

$$\int_a^x f(t)(x-t)\,dt.$$

So,

$$y(x) = -\int_a^x \Big(\underbrace{A(t) + (x-t)\big[B(t) - A'(t)\big]}_{K(x,t)}\Big)\,y(t)\,dt$$

$$\underbrace{+\int_a^x (x-t)\,g(t)\,dt + [A(a)y_0 + y_0'](x-a) + y_0}_{f(x)}.$$

$$\Rightarrow \quad y(x) = f(x) + \int_a^x K(x,t)\,y(t)\,dt.$$

Volterra equation. (2).

example: linear oscallator.

$$y'' + w^2 y = 0.$$

$$y(0) = 0 \qquad y'(0) = 1$$

$$\Rightarrow \quad A(x) = 0 \qquad B(x) = w^2 \qquad g(x) = 0$$

$$\Rightarrow \quad y(x) = x + w^2\int_0^x (t-x)\,y(t)\,dt.$$

$$y(x) = \frac{1}{\omega}\sin(\omega x)$$

boundary condition:

$$y(0) = 0, \qquad y(b) = 0.$$

$$y'' + \omega^2 y = 0.$$

$$y'(x) = -\int_0^x \omega^2 y\,dt + y'(0)$$

$$y = -\omega^2 \int_0^x (x-t)\,y(t)\,dt + y'(0)\cdot x.$$

use $y(b) = 0$.

$$\omega^2 \int_0^b (b-t)\,y(t)\,dt = b\cdot y'(0).$$

$$y = -\omega^2 \int_0^x (x-t)\,y(t)\,dt + \frac{\omega^2}{b}\cdot x \int_0^b (b-t)\,y(t)\,dt.$$

$$\int_0^x + \int_x^b$$

$$=$$

$$(x-t) = \boxed{\frac{x}{b}(b-t)} - \frac{t}{b}(b-x)$$

$$y = -\omega^2 \int_0^x (x-t)\,y(t)\,dt + \omega^2 \left(\int_0^x + \int_x^b\right)\left[(x-t) - \frac{t}{b}(b-x)\right] y(t)\,dt$$

$$= \underbrace{\omega^2 \int_0^x \frac{t}{b}(b-x)\,y(t)\,dt + \omega^2 \int_x^b \frac{x}{b}(b-t)\,y(t)\,dt}_{K(x,t)}.$$

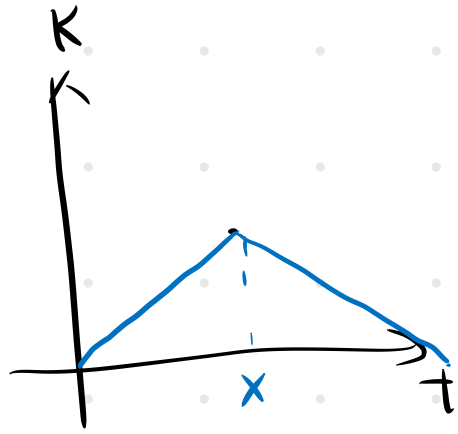
$$K(x,t) = \begin{cases} \frac{t}{b}(b-x), & t < x \\ \frac{x}{b}(b-t), & t > x. \end{cases}$$

$$y(x) = \omega^2 \int_0^b K(x,t)\,y(t)\,dt.$$

1. symmetric. $K(x,t) = K(t,x)$

2. continuous

3. $\frac{\partial K}{\partial t}$ is discontinuous.



Initial condition $\rightarrow$ Volterra

Boundary condition $\rightarrow$ Fredholm.

differential $\rightleftarrows$ integral.

Some special methods

1. Integral transform.

$$f(x) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{+\infty} e^{ixt} \varphi(t)\, dt. \qquad \text{Fourier.}$$

$$\varphi(x) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{+\infty} e^{-ixt} f(x)\, dt$$

$$f(x) = \int_0^{+\infty} e^{xt} \varphi(t)\, dt. \qquad \text{Laplace.}$$

$$\varphi(x) = \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} e^{xt} f(t)\, dt.$$

$$f(x) = \int_0^{+\infty} t^{x-1} \varphi(t)\, dt \qquad \text{Mellin}$$

$$\varphi(x) = \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} x^{-t} f(t)\, dt$$

$$f(x) = \int_0^{+\infty} t\varphi(t) J_0(xt)\, dt \qquad \text{Hankel}$$

$$\varphi(x) = \int_0^{+\infty} t f(t) J_0(xt)\, dt$$

Fourier Convolution Theorem.

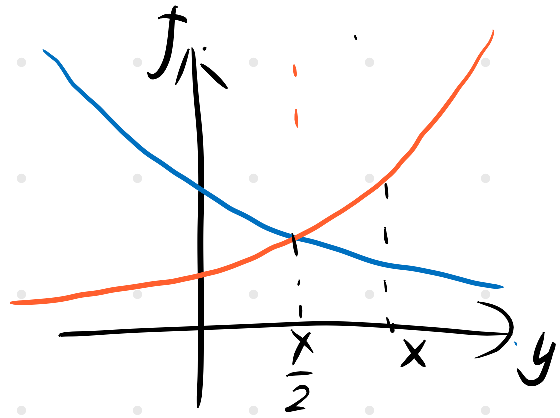
$$(f * g)(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \underline{g(y)}\, \underline{f(x-y)}\, dy.$$

3-D.

$$(f * g)(\vec{r}) = \frac{1}{(2\pi)^{3/2}} \int g(\vec{r}\,')\, f(\vec{r} - \vec{r}\,')\, d^3 r'$$

Faltung: folding.

example $f(y) = e^{-y}$, $f(x-y) = e^{-(x-y)}$



$$(f * g)^{T}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx\, e^{itx} \left[\int_{-\infty}^{+\infty} \underline{g(y)}\, \underline{f(x-y)}\, dy. \right]$$

$$= \underline{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dy\, g(y) \cdot e^{ity}} \cdot \underline{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx\, f(x-y)\, e^{it(x-y)}}$$

$$= G(t)\, F(t).$$

Inverse.

$$\frac{1}{\sqrt{2\pi}} \boxed{\int_{-\infty}^{+\infty} G(t)\, F(t) \cdot e^{-ixt}\, dt} = (f * g)(x)$$

$$= \frac{1}{\sqrt{2\pi}} \boxed{\int_{-\infty}^{+\infty} g(y)\, f(x-y)\, dy.}$$

Laplace Convolution Theorem

$$f_1(s) = \mathcal{L}\{F_1(t)\}$$

$$f_2(s) = \mathcal{L}\{F_2(t)\}$$

$$\underline{f_1(s)\cdot f_2(s)} = \int_0^{+\infty} e^{-sx} F_1(x)\,dx \int_0^{+\infty} e^{-sy} F_2(y)\,dy$$

let $t = x+y$. $\quad 0 \le y \le +\infty$

$y = y$. $\quad 0 \le x \le +\infty \quad 0 \le t-y \le +\infty$

$\boxed{y \le t}$

$$= \int_0^{+\infty} e^{-st}\,dt \boxed{\int_0^{t} F_1(t-y)\, F_2(y)\, dy}$$

$$= \mathcal{L}\left\{\int_0^{t} F_1(t-y) F_2(y)\,dy\right\}$$

$$= \mathcal{L}\{\underline{F_1 * F_2}\}$$

example. Fourier Transform Solution

$$f(x) = \int_{-\infty}^{+\infty} k(x-t)\,\varphi(t)\,dt$$

$$f(x) = \int_{-\infty}^{+\infty} K(\omega)\,\Phi(\omega)\, e^{-i\omega x}\, d\omega$$

$$\frac{1}{\sqrt{2\pi}}\ \underbrace{\boxed{\frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)\, e^{i\omega x}\, dx}}_{F(\omega)} = K(\omega)\,\Phi(\omega)$$

$$\Phi(\omega) = \frac{1}{\sqrt{2\pi}} \cdot \frac{F(\omega)}{K(\omega)}$$

$$\varphi(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{F(\omega)}{K(\omega)}\, e^{-i\omega x}\, d\omega$$

example Generalized Abel Equation.

$$f(x) = \int_0^x \frac{\varphi(t)}{(x-t)^\alpha}\,dt \qquad 0<\alpha<1.$$

Taking the Laplace transform.

$$\mathcal{L}\{f(x)\} = \mathcal{L}\left\{\int_0^x \boxed{\frac{\varphi(t)}{(x-t)^\alpha}}\,dt\right\} = \mathcal{L}\{x^{-\alpha}\}\,\mathcal{L}\{\varphi(x)\} \qquad \frac{s^{1-\alpha}}{\Gamma(1-\alpha)}$$

$$\frac{1}{s}\mathcal{L}\{\varphi(x)\} = \frac{\boxed{s^{-\alpha}}}{\Gamma(1-\alpha)}\mathcal{L}\{f(x)\}.$$

$$\Downarrow$$

$$\mathcal{L}\left\{\int_0^x \varphi(t)\,dt\right\} = \frac{\left(\mathcal{L}\{x^{\alpha-1}\}\,\mathcal{L}\{f(x)\}\right)}{\boxed{\Gamma(1-\alpha)\cdot\Gamma(\alpha)}}.$$

$$\mathcal{L}\{\varphi'(t)\} = \qquad = \frac{\sin\pi\alpha}{\pi}\,\mathcal{L}\left\{\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}\,dt\right\}.$$

$$\int_0^{+\infty} e^{-st}\frac{d\varphi}{\cancel{dt}}\cancel{dt}$$

$$= e^{-st}\cdot\varphi\Big|_0^{+\infty} + s\,\mathcal{L}\{\varphi(t)\}$$

$$\varphi(x) = \left[\int_0^x \varphi(t)\,dt\right]'$$

$$\boxed{\varphi(t) = \frac{\sin\pi\alpha}{\pi}\frac{d}{dx}\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}\,dt}$$

Generating Function Method.

orthonomal

$$f(x) = \int_{-1}^{+1} \frac{\varphi(t)}{\sqrt{1-2xt+x^2}}\, dt, \quad -1 \le x \le 1$$

Legendre polynomials.

$$\frac{1}{\sqrt{1-2xt+x^2}} = \sum_{n=0}^{\infty} P_n(t)\, x^n.$$

$$\varphi(t) = \sum_{m=0}^{\infty} a_m P_m(t).$$

$$f(x) = \sum_{m,n} a_m x^n \int_{-1}^{1} P_n(t) P_m(t).$$

$$= \sum_{n=0}^{\infty} a_n x^n \cdot \frac{2}{2n+1} = \sum_{n=0}^{\infty} \frac{2a_n}{2n+1} x^n$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

equate them.

$$\frac{f^{(n)}(0)}{n!} = \frac{2a_n}{2n+1}$$

$$a_n = \frac{2n+1}{2} \cdot \frac{f^{(n)}(0)}{n!}$$

$$\varphi(t) = \sum_{n=0}^{\infty} \frac{2n+1}{2} \cdot \frac{f^{(n)}(0)}{n!} x^n.$$

Separable Kernel.

$$K(x,t) = \sum_{j=1}^{n} M_j(x) N_j(t). \quad n < +\infty.$$

degenerate.

$$K(x,t) = \cos(t-x).$$
$$= \cos t \cos x + \sin t \sin x.$$

For a fredholm equation (2)

$$\varphi(x) = \underline{f(x)} + \lambda \sum_{j=1}^{n} M_j(x) \underbrace{\int_a^b N_j(t)\varphi(t)\,dt}_{c_j}.$$

So, $\underline{\int_a^b N_i(x)\varphi(x)} = \int_a^b \Big[f(x) + \lambda \sum_{j=1}^{n} c_j M_j(x)\Big] N_i(x)$

$$\underbrace{\int_a^b N_i(x)\varphi(x)\,dx}_{c_i} = \underbrace{\int_a^b f(x) N_i(x)\,dx}_{b_i} + \lambda\sum_{j=1}^{n} c_j \underbrace{\int_a^b M_j N_i(x)\,dx}_{a_{ij}}$$

$$c_i = b_i + \lambda \sum_j a_{ij} c_j.$$

$$\vec{c} = \vec{b} + \lambda \mathbb{A} \vec{c}.$$

$$\underline{\vec{b}} = (\mathbb{1} - \lambda \mathbb{A})\vec{c}. \qquad \boxed{\vec{c} =} (\mathbb{1} - \lambda\mathbb{A})^{-1}\vec{b}$$

$$\det(\mathbb{1} - \lambda \mathbb{A}) = 0.$$

example.

$$\varphi(x) = \lambda \int_{-1}^{1} \underline{(t+x)}\,\varphi(t)\,dt.$$

$$t + x = \underbrace{1}_{M_1(x)} \cdot \underbrace{t}_{N_1} + \underbrace{x}_{M_2} \cdot \underbrace{1}_{N_2}.$$

$$a_{ij} = \int_{-1}^{1} N_i M_j \,dx$$

$\frac{1}{2}x^2 t.$

$$a_{11} = \int_{-1}^{1} x\,dx = 0, \qquad a_{22} = 0, \qquad a_{12} = \int_{-1}^{1} tx\,dx = \frac{1}{2}x^2 t\Big|$$

$$|\mathbb{1} - \lambda \mathbb{A}| = 0 \Rightarrow \begin{vmatrix} 1 & -\frac{2\lambda}{3} \\ -2\lambda & 1 \end{vmatrix} = 0 \Rightarrow \lambda = \pm \frac{\sqrt{3}}{2}.$$

$$1 = \frac{4\lambda^2}{3}$$

$$\vec{b} = (\mathbb{1} - \lambda \mathbb{A})\vec{c} \Rightarrow$$

$$\begin{pmatrix} 1 & \mp \frac{\sqrt{3}}{3} \\ \mp\sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow c_1 \mp \frac{\sqrt{3}}{3} c_2 = 0$$

let $c_1 = 1$

$\varphi_1(x) = \frac{\sqrt{3}}{2}(1+3x) \rightarrow \lambda = +$

$\varphi_2(x) = -\frac{\sqrt{3}}{2}(1-\sqrt{3}x) \quad \lambda -$

Neumann Series.

$$\varphi(x) = f(x) + \lambda \int_a^b K(x,t)\,\varphi(t)\,dt. \quad f(x) \neq 0.$$

$$\varphi(x) \approx \varphi_0(x) = f(x)$$

$$\varphi_1(x) = f(x) + \lambda \underbrace{\int_a^b K(x,t) f(t)\,dt}_{u_1}.$$

$$\varphi_2(x) = f(x) + \lambda \int_a^b K(x,t) f(t_1)\,dt_1$$

$$+ \lambda^2 \underbrace{\int_a^b \int_a^b K(x_1, t_1) K(x_2, t_2) f(t_2)\,dt_2\,dt_1}_{u_2}.$$

$\vdots$

$$\varphi_n(x) = \sum_{i=0}^{n} \lambda^i u_i(x).$$

$u_0 = f(x)$

$u_1 = \int_a$

$$u_n(x) = \underbrace{\int_a^b\int\int \cdots \int_a^b}_{n} \left[\prod_{i=1}^{n} K_i\right] f(t_n)\, dt_n \cdots dt_1$$

we expect.

$$\varphi(x) = \lim_{n\to+\infty} \varphi_n(x) = \lim_{n\to+\infty} \sum_{i=0}^{n} \lambda^i u_i(x).$$

$$|\lambda^n u_n(x)| \le |\lambda^n| \boxed{|f|_{max}} |K|_{max}^n \cdot |b-a|^n$$

$$\lambda \cdot |K|_{max} |b-a| < 1 \quad \text{converge.}$$

operator form.

$$\varphi = \lambda \hat{K} \varphi + f$$

$$\hat{K} = \int_a^b K(x,t)[\;]\, dt.$$

$$\varphi = (\mathbb{1} - \lambda \hat{K})^{-1} f.$$

Binomial expansion

example.

$$\varphi(x) = x + \frac{1}{2}\int_{-1}^{1} (t-x)\varphi(t)\, dt.$$

$$\varphi_0 = x.$$

$$\varphi_1 = x + \frac{1}{2}\int_{-1}^{1} (t-x)\, t\, dt = x + \frac{1}{3}.$$

$$\varphi_2 = x + \frac{1}{2}\int_{-1}^{1} (t-x)\left(x + \frac{1}{3}\right) dt = x + \frac{1}{3} - \frac{x}{3}.$$

$$\varphi_3 = x + \frac{1}{2}\int_{-1}^{1} (t-x)\left(x + \frac{1}{3} - \frac{x}{3}\right) dt = x + \frac{1}{3} - \frac{x}{3} - \frac{1}{3^2}.$$

$$\varphi(x) = \frac{3}{4}x + \frac{1}{4}.$$

Hilbert-Schmidt Theory.

Symmetrization of Kernel.

$$K(x,t) = K(t,x)$$

$$\boxed{\sqrt{\rho(x)}\,\varphi(x)}\,\sqrt{\rho(x)} = f(x) + \lambda \int_a^b \boxed{K(x,t)}\,\boxed{\rho(t)}\,\varphi(t)\,\sqrt{\rho(x)} \cdot dt.$$

$$\psi(x) = \sqrt{\rho(x)}\, f(x) + \lambda \int \boxed{\left[K(x,t)\sqrt{\rho(x)\rho(t)}\,\right]}\, \psi(t)\, dt$$

Orthogonal Eigenfunction.

$$\varphi(x) = \lambda \int_a^b \boxed{K(x,t)}\,\varphi(t)\, dt.$$

$$\hat{K}\varphi(x) = \frac{1}{\lambda}\varphi(x)$$

$$\langle \psi | \varphi \rangle = \int_a^b \psi^*(x)\,\varphi(x)\, dx.$$

$$\langle \psi | \hat{K}\varphi \rangle = \int_a^b \psi^*(x)\left[\int_a^b K(x,t)\,\varphi(t)\, dt\right] dx$$

$$= \int_a^b dt \int_a^b dx \left[K(t,x)\,\psi(x)\right]^* \varphi(t).$$

$$= \langle \hat{K}\psi | \varphi \rangle.$$

Inhomogeneous Integral function.

$$\varphi(x) = f(x) + \lambda \int_a^b K(x,t)\varphi(t)\,dt.$$

$$\varphi_n(x) = \lambda_n \int_a^b K(x,t)\,\varphi_n(t)\,dt.$$

$$\varphi(x) = \sum_{n=1}^{\infty} a_n \varphi_n(x).$$

$$f(x) = \sum_{n=1}^{\infty} b_n \varphi_n(x).$$

$$\sum_{n=1}^{\infty} a_n \varphi_n(x) = \sum_{n=1}^{\infty} b_n \varphi_n(x) + \lambda \boxed{\sum a_n \int_a^b K(x,t)\varphi_n(t)\,dt}$$

$$\sum_{n=1}^{\infty} a_n \varphi_n(x) = \sum_{n=1}^{\infty} b_n \varphi_n(x) + \lambda \sum_{n=1}^{\infty} \frac{a_n \cdot \varphi_n}{\lambda_n}.$$

$$a_i = b_i + \lambda \frac{a_i}{\lambda_i}$$

$$a_i = b_i + \frac{\lambda}{\lambda_i - \lambda} b_i.$$

$$\varphi(x) = f(x) + \lambda \sum_{i=1}^{\infty} \frac{\varphi_i(x)}{\lambda_i - \lambda} \int_a^b f(t)\,\varphi_i(t)\,dt.$$

example.

$$\varphi(x) = x^3 + 1 \int_{-1}^{1} (t+x)\varphi(t)\,dt.$$

$$\lambda_1 = \frac{\sqrt{3}}{2} \qquad \lambda_2 = -\frac{\sqrt{3}}{2}$$

$$\varphi_1 = \frac{\sqrt{3}}{2}\left(x + \frac{1}{\sqrt{3}}\right) \qquad \varphi_2 = \frac{\sqrt{3}}{2}\left(x - \frac{1}{\sqrt{3}}\right).$$

$$\varphi(x) = x^3 + \sum \frac{\varphi_i(x)}{\lambda_i - 1} \int_{-1}^{1} t^3 \varphi_i(t)\,dt$$

$$\varphi(x) = x^3 + \frac{\frac{\sqrt{3}}{2}\left(x+\frac{1}{\sqrt{3}}\right)}{\frac{\sqrt{3}}{2}-1} \cdot \frac{\sqrt{3}}{2}\int_{-1}^{1} t^3\left(t+\frac{1}{\sqrt{3}}\right)dt$$

$$+ \frac{\frac{\sqrt{3}}{2}\left(x+\frac{1}{\sqrt{3}}\right)}{-\frac{\sqrt{3}}{2}-1} \cdot \frac{\sqrt{3}}{2}\int_{-1}^{1} t^3\left(t-\frac{1}{\sqrt{3}}\right)dt.$$

$$= x^3 - \frac{6}{5}(2x+1)$$