Chapter 22

Calculus of Variations.

$$\frac{d}{dx}\Big|_{x=0}(x^2) = 0.$$

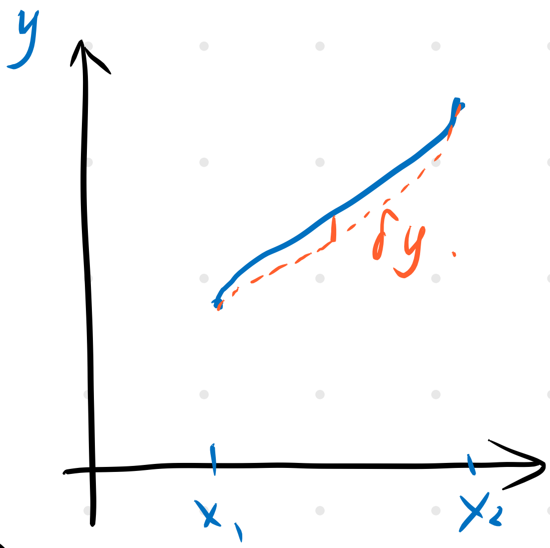
$$\int_a^b (\qquad)\, dx$$

$$J[\underline{y}] = \int_{x_1}^{x_2} f\left(y(x), \frac{dy(x)}{dx}, x\right) dx.$$

$$\delta J[y] = \delta \int_{x_1}^{x_2} f\left(y(x), \frac{dy(x)}{dx}, x\right) dx.$$

$$y(x, \alpha) = \boxed{y(x, 0)} + \alpha\, \eta(x).$$

$$\boxed{\eta(x_1) = \eta(x_2) = 0}$$



$$\frac{\partial J[y]}{\partial \alpha}\Big|_{\alpha=0} = 0$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y_x} \cdot \frac{\partial y_x}{\partial \alpha}\right] dx = 0$$

$$\frac{\partial y}{\partial \alpha} = \eta(x). \qquad \frac{\partial y_x}{\partial \alpha} = \frac{\partial^2 y}{\partial \alpha\, \partial x} = \frac{d\eta(x)}{dx}.$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} \cdot \eta(x) + \underline{\frac{\partial f}{\partial y_x} \cdot \frac{d\eta(x)}{dx}}\right] dx = 0$$

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y_x} \frac{d\eta(x)}{dx}\, dx = \cancel{\frac{\partial f}{\partial y_x} \cdot \eta(x)\Big|_{x_1}^{x_2}} - \int_{x_1}^{x_2} \eta(x) \underline{\frac{d}{dx}} \frac{\partial f}{\partial y_x}\, dx$$

$$\int_{x_1}^{x_2} \boxed{\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y_x}\right]} \eta(x)\, dx = 0 = \frac{\partial J}{\partial \alpha}$$

$$\Rightarrow \quad \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y_x} = 0.$$

Euler - Lagrange function.

example.

$$J = \int_{x_1 y_1}^{x_2 y_2} ds = \int \sqrt{dx^2 + dy^2}$$

$$= \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$f(y, y_x, x) = \sqrt{1 + y_x^2}$$

$$\frac{d}{dx}\left(\frac{y_x^2}{\sqrt{1+y_x^2}}\right) = 0$$

$$\frac{y_x^2}{\sqrt{1+y_x^2}} = C. \Rightarrow y_x = a.$$

$$y = kx + m.$$

example Optical Path Near A Black Hole

Suppose. $v(y) = \frac{y}{b}$.

$y = 0 \quad v = 0 \rightarrow$ event horizon.

$$\Delta t = \int dt = \int \frac{ds}{v} = \int \frac{b}{y}\sqrt{dx^2 + dy^2} \rightarrow \text{minimum}$$

$$= b \int \frac{\sqrt{x_y^2 + 1}}{y}\, dy$$

$$f(x, x_y, y) = \frac{\sqrt{x_y^2+1}}{y}$$

E-L

$$\frac{\partial f}{\partial x} - \frac{d}{dy}\frac{\partial f}{\partial x_y} = -\frac{d}{dy}\left(\boxed{\frac{x_y}{y\sqrt{1+x_y^2}}}\right) = 0$$

$$\frac{x_y}{y\sqrt{1+x_y^2}} = C_1$$

$$x_y^2 = C_1^2 y^2 (1+x_y^2)$$

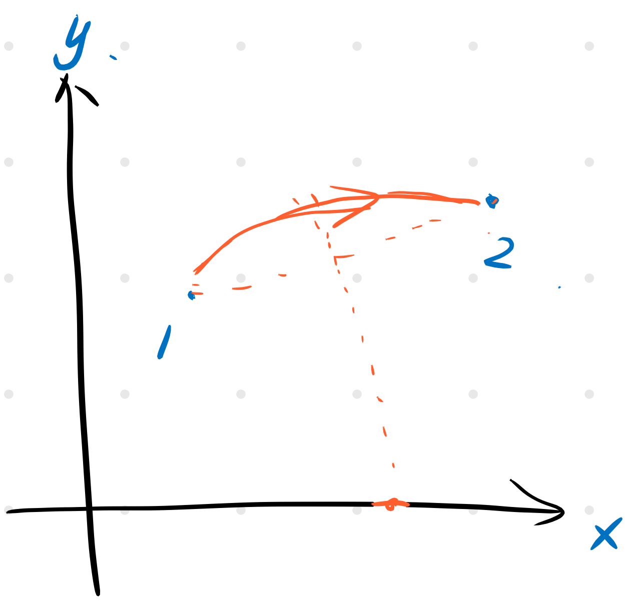
$$x_y^2(1 - C_1 y^2) = C_1^2 y^2$$

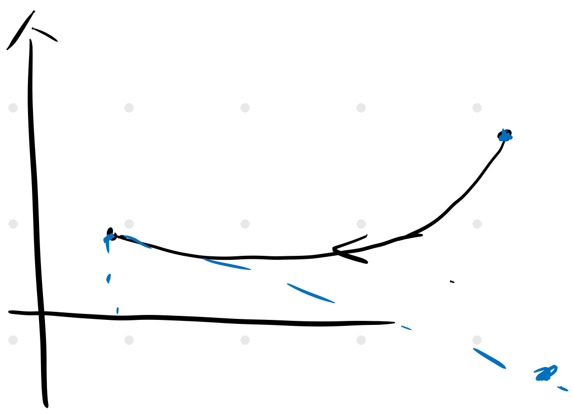
$$\int^x dx = \int^y \frac{C_1 y \, dy}{\sqrt{1 - C_1 y^2}}$$

$$x + C_2 = \frac{\sqrt{1 - C_1^2 y^2}}{C_1}$$

$$(x + C_2)^2 + y^2 = \frac{1}{C_1^2}$$

Arc.





Alternate of E-L Equation

$$\frac{\partial f}{\partial x} - \frac{d}{dx}\left(\underline{f - y_x \frac{\partial f}{\partial y_x}}\right) = 0$$

$$\cancel{\frac{\partial f}{\partial x} - \frac{\partial f}{\partial x}} - \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} - \cancel{\frac{\partial f}{\partial y_x}\frac{\partial y_x}{\partial x}}$$

$$+ \cancel{\left(\frac{d}{dx} y_x\right)\frac{\partial f}{\partial y_x}} + y_x \frac{d}{dx}\frac{\partial f}{\partial y_x} = 0 .$$

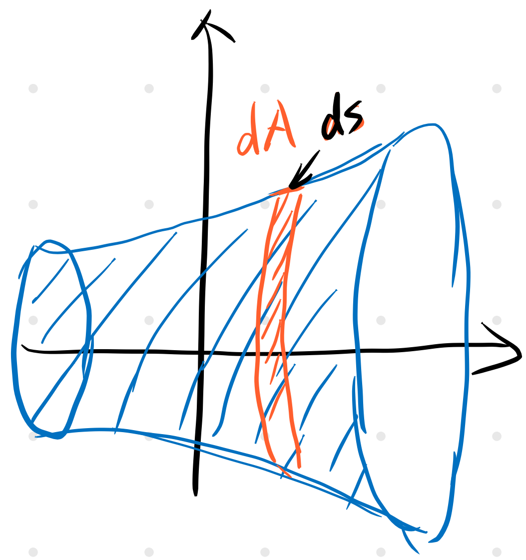
$$\frac{d}{dx}\frac{\partial f}{\partial y_x} - \frac{\partial f}{\partial y} = 0 .$$

$$\Downarrow$$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y_x} = 0 .$$

if $f(y, y_x)$, $f - y_x \frac{\partial f}{\partial y_x} = \text{Const}$.

example: Soap Film.



$$dA = 2\pi y\, ds = 2\pi y \sqrt{1+y_x^2}\, dx .$$

$$f(y, y_x, x) = 2\pi y \sqrt{1+y_x^2}$$

$$f - y_x \frac{\partial f}{\partial y_x} = \text{Const} .$$

$$y\sqrt{1+y_x^2} - y_x \cdot \frac{y \cdot y_x}{\sqrt{1+y_x^2}} = C_1$$

$$y = C_1 \sqrt{1+y_x^2}$$

$$\frac{y}{\sqrt{1+y_x^2}} = C_1$$

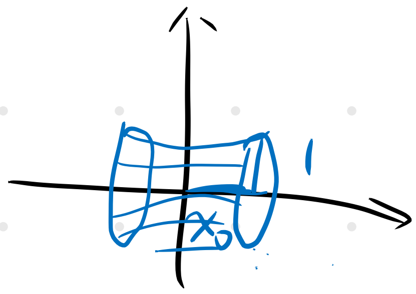
$$\frac{y^2}{1+y_x^2} = C_1^2 .$$

$$\frac{dy}{dx} = \frac{1}{C_1}\sqrt{y^2 - C_1^2} .$$

$$x = c_1 \operatorname{arccosh}\left(\frac{y}{c_1}\right) + c_2$$

$$y = c_1 \cosh\left(\frac{x - c_2}{c_1}\right).$$ catenoid.

Consider. $(-x_0, 1)$ $(x_0, 1)$

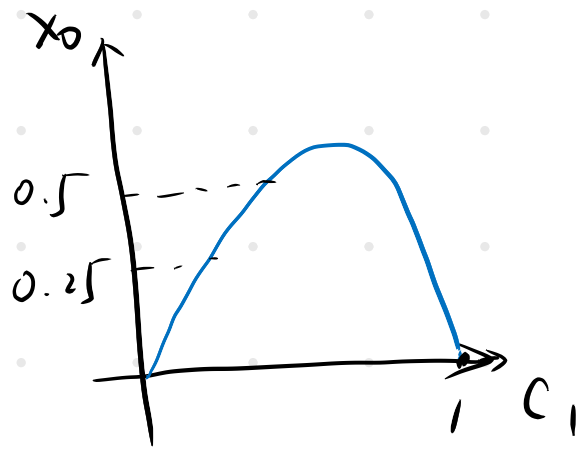


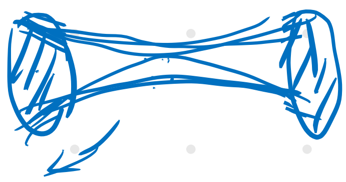
$$y = c_1 \cosh\left(\frac{x}{c_1}\right)$$

$$y(x_0) = c_1 \cosh\left(\frac{x_0}{c_1}\right) = 1 .$$

$$\boxed{x_0 = 1}$$ c_1

$$x_0 = \frac{1}{2}.$$



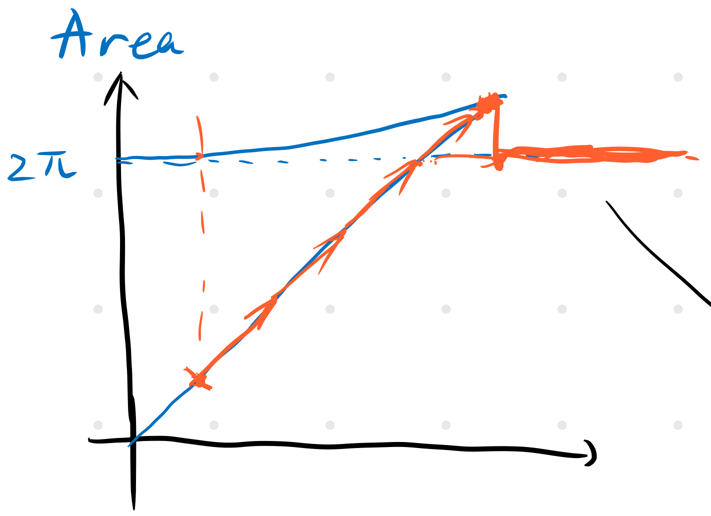


$A = 2\pi$.

$$A = 4\pi \int_0^y y\sqrt{1+y_x^2}\, dx = \frac{4\pi}{c_1} \int_0^{x_0} y^2\, dx$$

$$= 4\pi c_1 \int_0^{x_0} \left(\cosh \frac{x}{c}\right)^2 dx .$$

$$= \pi c_1^2 \left[\sinh\left(\frac{2x_0}{c_1}\right) + \frac{2x_0}{c_1} \right]$$



Goldschmidt discontinous solution.

$$J = \int_{x_1}^{x_2} f\,(u_1(x), \cdots u_n(x), u_{1x}(x) \cdots u_{nx}(x), x)\,dx$$

$$u_i(x,\alpha) = \underline{u_i(x,0)} + \alpha\,\eta_i(x)$$

$$\left.\frac{\partial J}{\partial \alpha}\right|_{\alpha=0} = 0 \Rightarrow \int_{x_1}^{x_2} \sum_i \left[\frac{\partial f}{\partial u_i}\eta_i + \underline{\frac{\partial f}{\partial u_{ix}}\boxed{\eta_{ix}}}\right] dx = 0$$

$$\Rightarrow \int_{x_1}^{x_2} \underline{\sum_i \left[\frac{\partial f}{\partial u_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial u_{ix}}\right)\right]\eta_i}\ dx = 0$$

$$\frac{\partial f}{\partial u_i} - \frac{d}{dx}\frac{\partial f}{\partial u_{ix}} = 0, \quad i = 1, 2 \cdots n.$$

Hamilton's Principle

$$\mathcal{L} := T - V$$

$x \to t$
$y_i \to x_i(t)$
$y_{ix} \to \dot{x}_i(t)$

$$\boxed{\delta \int_{t_1}^{t_2} \mathcal{L}(x_1 \cdots \dot{x}_1 \cdots t)\,dt}$$

By E-L equation

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_i} - \frac{\partial \mathcal{L}}{\partial x_i} = 0.$$

$\downarrow$ Lagrangian equation of motion

$$\dot{p}_i = \frac{\partial \mathcal{L}}{\partial q_i}$$

example. moving 1 particle in cartesian

$$\mathcal{L} = \frac{1}{2} m\dot{x}^2 - V.$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} \qquad \frac{\partial \mathcal{L}}{\partial x} = -\frac{dV}{dx}.$$

$$\frac{d}{dx}(m\dot{x}) = F(x)$$

In polar coordinates.

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m(\dot{\rho}^2 + \rho^2\dot{\varphi}^2).$$

$$V = 0.$$

ρ, φ.

$$\begin{cases} \frac{d}{dt}(m\dot{\rho}) = m\rho\dot{\varphi}^2 \rightarrow \\ \frac{d}{dt}\left(\boxed{m\rho^2\dot{\varphi}}\right) = 0 \end{cases}$$

angular momentum.

canonical momentum. $p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$

So $$d\mathcal{L} = \sum_i \left(\frac{\partial \mathcal{L}}{\partial q_i} dq_i + \boxed{\frac{\partial \mathcal{L}}{\partial \dot{q}_i}} d\dot{q}_i \right) + \frac{\partial \mathcal{L}}{\partial t} dt.$$

$$d\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \dot{q}_i\right) - \dot{q}_i d\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)$$

$$d\left(\mathcal{L} - \sum_i p_i \dot{q}_i\right) = \sum_i \left(\underbrace{\boxed{\frac{\partial \mathcal{L}}{\partial q_i}}}_{\dot{p}_i} dq_i - \dot{q}_i dp_i \right) + \frac{\partial \mathcal{L}}{\partial t} dt.$$

$$d\left(\mathcal{L}-\sum_i p_i \dot{q}_i\right) = \sum_i \left(\boxed{\frac{\partial \mathcal{L}}{\partial q_i}} dq_i - \dot{q}_i\, dp_i\right) + \frac{\partial \mathcal{L}}{\partial t} dt$$

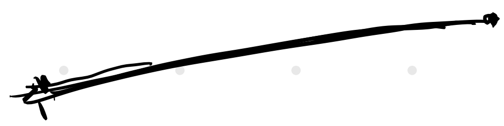
$$\dot{p}_i$$

$$d\Big(\underbrace{\boxed{\sum_i p_i \dot{q}_i - \mathcal{L}}}_{H}\Big) = \sum_i \left(\dot{q}_i\, dp_i - \dot{p}_i\, dq_i\right) - \frac{\partial \mathcal{L}}{\partial t} dt$$

$$dH = \sum\left(\frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i\right) + \frac{\partial H}{\partial t} dt$$

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \qquad \frac{\partial H}{\partial q_i} = -\dot{p}_i \qquad \frac{\partial H}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}.$$

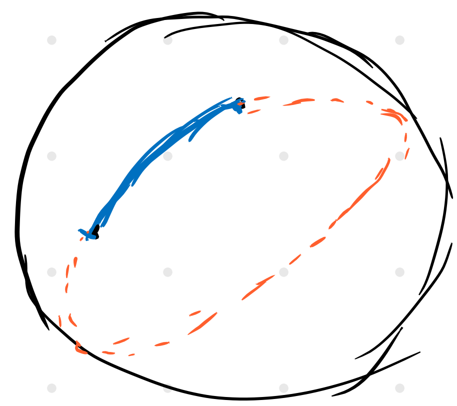
Geodesics.



ds $\qquad dq^i$

contravariant ↑.

covariant ↓.



$$ds^2 = \boxed{g_{ij}}\, dq^i\, dq^j$$

$$ds^2 = dx^2 + dy^2 + dz^2.$$

$$\boxed{dq^i dq^j = 0.\ (i \neq j)}$$

$q(u)$.

$$J = \int_A^B \frac{ds}{du} du = \int \frac{\sqrt{g_{ij}\, dq^i dq^j}}{du} du$$

$$= \int_A^B \sqrt{g_{ij}\, \dot{q}^i \dot{q}^j}\, du$$

$$\mathcal{L} = \frac{m}{2} g_{ij} \dot{q}^i \dot{q}^j$$

$(a\tau + b)$

affine transform.

$$\delta \int_A^B g_{ij} \dot{q}^i \dot{q}^j = 0.$$

$$\frac{\partial g_{ij} \dot{q}^i \dot{q}^j}{\partial q^k} - \frac{d}{du} \frac{\partial g_{ij} \dot{q}^i \dot{q}^j}{\partial \dot{q}^k} = 0.$$

$$= \frac{\partial g_{ij}}{\partial q^k} \cdot \dot{q}^i \dot{q}^j - \frac{d}{du}\left(g_{ij} \frac{\partial \dot{q}^i \dot{q}^j}{\partial \dot{q}^k} \right).$$

$$= \frac{\partial g_{ij}}{\partial q^k} \dot{q}^i \dot{q}^j - \frac{d}{du}\left(g_{kj} \dot{q}^j + g_{ik} \dot{q}^i \right).$$

$$\frac{d \dot{q}^j}{du} = \ddot{q}^j \qquad \frac{d g_{kj}}{du} = \frac{\partial g_{kj}}{\partial q^i} \cdot \dot{q}^i$$

$$\frac{1}{2} \dot{q}^i \dot{q}^j \left[\frac{\partial g_{ij}}{\partial q^k} - \frac{\partial g_{kj}}{\partial q^i} - \frac{\partial g_{ik}}{\partial q^j} \right] - g_{ik} \ddot{q}^i = 0.$$

g^{kl}. $\quad g^{kl} g_{ik} = \delta_i^l$

$$\boxed{\frac{d^2 q^l}{du^2} + \frac{d q^i}{du} \frac{d q^j}{du} \cdot \frac{1}{2} g^{kl} \left[-\frac{\partial g_{ij}}{\partial q^k} + \frac{\partial g_{kj}}{\partial q^i} + \frac{\partial g_{ik}}{\partial q^j} \right] = 0}$$

By substuting Christoffel symbol.

$$\Gamma^n_{ij} = g^{nk} [ij, k] = \frac{1}{2} g^{nk} \left[\frac{\partial g_{ik}}{\partial q^j} + \frac{\partial g_{jk}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^k} \right].$$

$$\boxed{\frac{d^2 q^l}{du^2} + \frac{d q^i}{du} \cdot \frac{d q^j}{du} \Gamma^l_{ij} = 0}.$$

Constrained extrema.

$$f(x,y,z) \qquad \underline{g(x,y,z)=0}$$

$$dg = \left(\frac{\partial g}{\partial x}\right)_{yz} dx + \left(\frac{\partial g}{\partial y}\right)_{xz} dy + \left(\frac{\partial g}{\partial z}\right)_{xy} dz = 0$$

$$dz = \left(\frac{\partial z}{\partial x}\right)_y \underline{dx} + \left(\frac{\partial z}{\partial y}\right)_x \underline{dy}$$

$$\left(\frac{\partial g}{\partial x}\right)_{yz} + \left(\frac{\partial g}{\partial z}\right)_{xy} \left(\frac{\partial z}{\partial x}\right)_y = 0$$

$$\underline{\left(\frac{\partial z}{\partial x}\right)_y} = - \frac{\left(\frac{\partial g}{\partial x}\right)_{yz}}{\left(\frac{\partial g}{\partial z}\right)_{xy}}$$

$$\left(\frac{\partial z}{\partial y}\right)_x = - \frac{\left(\frac{\partial g}{\partial y}\right)_{xz}}{\left(\frac{\partial g}{\partial z}\right)_{xy}}$$

$$\left(\frac{\partial f}{\partial x}\right)_y = \left(\frac{\partial f}{\partial x}\right)_{yz} + \left(\frac{\partial f}{\partial z}\right)_{xy} \left(\frac{\partial z}{\partial x}\right)_y$$

$$= \left(\frac{\partial f}{\partial x}\right)_{yz} - \underbrace{\boxed{\frac{\left(\frac{\partial f}{\partial z}\right)_{xy}}{\left(\frac{\partial g}{\partial z}\right)_{xy}}}}_{\lambda} \left(\frac{\partial g}{\partial x}\right)_{yz}$$

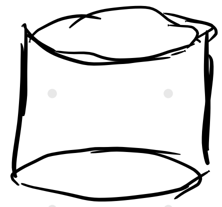
$$= \left(\frac{\partial f}{\partial x}\right)_{yz} - \lambda \left(\frac{\partial g}{\partial x}\right)_{yz} = 0.$$

$$\frac{\partial f}{\partial x_i} - \sum_j \lambda_j \frac{\partial g_j}{\partial x_i} = 0.$$

example.

$$V = \pi r^2 h$$

$$S = 2\pi(rh + r^2)$$



$$\frac{\partial S}{\partial r} - \lambda \frac{\partial V}{\partial r} = 2\pi(h + 2r) - \lambda\, 2\pi r h = 0$$

$$\frac{\partial S}{\partial h} - \lambda \frac{\partial V}{\partial h} = 2\pi r - \lambda\, \pi r^2 = 0.$$

$$\frac{h+2r}{\;} = \frac{2h}{\;} \Rightarrow h = 2r.$$

Variation with Constrains.

$$J = \int f\left(y_i, \frac{\partial y_i}{\partial x_j}, x_j\right) dx_j$$

$$\delta \int \lambda_k(x_j)\, \varphi_k\left(y_i, \frac{\partial y_i}{\partial x_j}, x_j\right) dx_j = 0.$$

$$\downarrow$$

$$\delta \int \Big[\underbrace{f\left(y_i, \frac{\partial y_i}{\partial x_j}, x_j\right) + \sum_k \lambda_k(x_j)\, \varphi_k\left(y_i, \frac{\partial y_i}{\partial x_j}, x_j\right)}_{= g\left(y_i, \frac{\partial y_i}{\partial x_j}, x_j\right)} \Big] dx_j = 0.$$

Lagrangian Formulation with Constrains.

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0.$$

$$\delta \int \left[\mathcal{L}(q_i, \dot{q}_i, t) + \sum_k \lambda_k(t)\,\varphi_k(q, t) \right] dt = 0.$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = \sum_k \lambda_k \underbrace{\left(\frac{\partial \varphi_k}{\partial q_i}\right)}_{a_{ik}}.$$

Simple Pendulum.

$$\varphi_1 = r - l = 0. \qquad r, \theta.$$

$$\mathcal{L} = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + mgr\cos\theta.$$

$$a_{r1} = \frac{\partial(r-l)}{\partial r} = 1 \qquad a_{\theta 1} = 0.$$

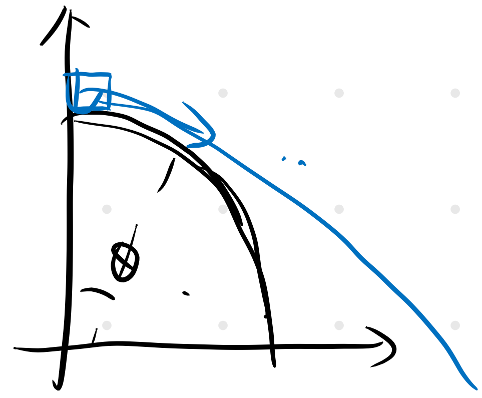
$$r: \begin{cases} \dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot{r}} - \dfrac{\partial \mathcal{L}}{\partial r} = \lambda_1 \\[2ex] \dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot{\theta}} - \dfrac{\partial \mathcal{L}}{\partial \theta} = 0. \end{cases}$$

$$\frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 - mg\cos\theta = \lambda_1$$

$$\boxed{\frac{d}{dt}(mr^2\dot{\theta}) + mgr\sin\theta = 0}$$

$$ml\dot{\theta}^2 + mg\cos\theta = -\lambda_1$$

example



$$\mathcal{L} = T - V = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - mgr\cos\theta .$$

$$\varphi_1 = r - l = 0 .$$

$$a_{r1} = \frac{\partial \varphi_1}{\partial r} = 1 \qquad a_{\theta 1} = \frac{\partial \varphi_1}{\partial \theta} = 0 .$$

$$\begin{cases} -ml\dot{\theta}^2 + mg\cos\theta = \lambda_1(\theta) \\ ml^2\ddot{\theta} - \underline{mgl\sin\theta} = 0 . \end{cases}$$

$$-2ml\ddot{\theta} - \underline{mg\sin\theta} = \frac{d\lambda_1(\theta)}{d\theta}$$

$$\frac{d\lambda(\theta)}{d\theta} = -3mg\sin\theta .$$

$$\lambda(\theta) = 3mg\cos\theta + C .$$

$$\underline{-ml\dot{\theta}^2|_{\theta=0}} + mg = 3mg + C .$$

$$C \leq -2mg .$$

$$\lambda(\theta) = mg(3\cos\theta - 2) . \quad \geq 0 .$$

$$\cos\theta = \frac{2}{3} .$$