13.Gamma Function

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1 13.1 Definition,Properties

In order to get the analytic continuation of factorial notation and analyse the properties, we will research the the gamma function in this chapter. The factorial is:

$$n! = 1 \times 2 \times 3 \times ... \times n$$

And first, here are three definitions of Gamma function which satisfies $\Gamma(n) = n!$ when n is an positive integer.

- Infinite limit(Euler)
- Definite integral(Euler)
- Infinite product(Weierstrass)

1.1 Infinite Limit

The first definition:

$$\Gamma(z) = \lim_{n \to \infty} \frac{1 \cdot 2 \cdot 3 ... n}{z(z+1)(z+2)...(z+n)} n^z, z \neq 0, -1, -2, = 3...$$

Difference equation:

$$\Gamma(z+1) = z\Gamma(z)$$

From the definition,

$$\begin{aligned}
\Gamma(1) &= 1 \\
\Gamma(2) &= 1 \\
\Gamma(3) &= 2\Gamma(2) = 2 \\
\Gamma(4) &= 6\Gamma(3) = 6
\end{aligned}$$

1.2 Definite Intergral

The second definition:

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, Re(z) > 0$$

An intersting result:

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}$$

An important function to show the equivalence of these two definitions:

$$F(z,n) = n^z \int_0^1 (1-u)^n u^{z-1}$$

1.3 Infinite Product

The third definition:

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} (1 + \frac{z}{n}) e^{-\frac{z}{n}}$$

1.4 Properties

- $\Gamma(z+1) = z\Gamma(z), \Gamma(z+1) = z!$
- $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(z\pi)}$
- $\Gamma(1+z)\Gamma(z+\frac{1}{2}) = 2^{-2z}\sqrt{\pi}\Gamma(2z+1)$
- Residues:

$$R_n = \lim_{\epsilon \to 0} \epsilon \Gamma(-n+\epsilon) = \frac{(-1)^n}{n!}$$

- Schlaefli Integral:

$$\int_C e^{-t}t^v dt = (e^{2\pi i\nu} - 1)\Gamma(\nu+1) = 2ie^{i\nu\pi}\Gamma(\nu+1)\sin(\nu\pi)$$

2 13.2 Digamma and Polygamma Function

2.1 Digamma Function

Definition:

$$\psi(z+1) = \frac{d}{dz} ln\Gamma(z+1) = \frac{[\Gamma(z+1)]'}{\Gamma(z+1)}$$

2.2 Polygamma Function

The digamma function may be differentiated repeatedly, giving rise to the polygamma function:

$$\psi^{(m)}(z+1) \equiv \frac{d^{m+1}}{dz^{m+1}} ln\Gamma(z+1)$$

Set z to be zero, defining the Riemann-ζ function,

$$\zeta(m) \equiv \sum_{n=1}^{\infty} \frac{1}{n^m}$$

2.3 Maclaurin Expansion

Maclaurin expansion for $\ln\Gamma(z+1)$

$$\ln\Gamma(z+1) = \sum_{n=1}^{\infty} \frac{z^n}{n!}\psi^{(n-1)}(1) = -\gamma z + \sum_{n=2}^{\infty} (-1)^n \frac{z^n}{n}\zeta(n)$$

3 13.3 The Beta Function

Then we research the product of the gamma functions.

3.1 Forms

Beta function:

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

Some useful description:

- $B(p,q) = 2\int_0^{\pi/2} \cos^{2p-1}\theta \sin^{2q-1}\theta d\theta$
- $B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt$
- $B(p,q) = \int_\infty^0 \frac{u^{p-1}}{(u+1)^{p+q}}$

3.2 Legendre Duplication Formula

We have:

$$\Gamma(z+1)\Gamma(z+\frac{1}{2}) = \frac{\sqrt{\pi}}{2^{2z}}\Gamma(2z+1)$$

And we can proof this easily when n is an integer.

4 13.4 Stirling's Series

A roughly formula:

$$\ln\Gamma(z+1) \sim z\ln z - z$$

A little precise formula:

$$\ln\Gamma(z+1) \sim \sqrt{2\pi n}(\frac{n}{e})^n$$

5 13.5 Riemann-ζ Function

The definition of $\zeta(z)$:

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$$

Integral representation:

$$\zeta(z) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1}}{e^t - 1}, Re(z) > 1$$

ζ-function reflection formula:

$$\Gamma(\frac{z}{2})\pi^{-\frac{z}{2}}\zeta(z) = \Gamma(\frac{1-z}{2})\pi^{-\frac{1-z}{2}}\zeta(1-z)$$

6 13.6 Other Ralated Function Additional Readings(*)

6.1 Incomplete Gamma Functions

From gamma function, we can define another two function

$$\gamma(a,x) = \int_0^x e^{-t}t^{a-1}dt, R(a) > 0$$
$$\Gamma(a,x) = \int_x^\infty e^{-t}t^{a-1}dt, R(a) > 0$$

Clearly, these two functions are related, for

$$\gamma(a,x) + \Gamma(a,x) = \Gamma(a)$$

6.2 Incomplete Beta Function

From beta function,

$$B_x(p,q) = \int_0^x t^{p-1}(1-t)^{q-1}dt$$

6.3 Exponential Integral

Exponential integral,

$$-Ei(-x) \equiv \int_x^\infty \frac{e^{-t}}{t}dt \equiv E_1(x) = \Gamma(0,x) = \lim_{a\to 0}[\Gamma(a) - \gamma(a,x)]$$

More related integral:

- $si(x) = -\int_x^\infty \frac{\sin t}{t} dt$
- $Ci(x) = -\int_x^\infty \frac{\cos t}{t} dt$
- $ci(x) = -\int_x^\infty \frac{dt}{\ln t}$

6.4 Error Function

The error function $erf(z)$ and the complementary error function $erfc(z)$ are defined by the integrals

$$efr(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} = \pi^{-\frac{1}{2}} \gamma(\frac{1}{2}, z^2)$$
$$erfc(z) = 1 - efr(z) = \int_z^\infty e^{-t^2} = \pi^{-\frac{1}{2}} \Gamma(\frac{1}{2}, z^2)$$