Chapter 19

Fourier Series

19.1 General Properties

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Coefficients:

$$\int_0^{2\pi} f(x) \cos(ix)\, dx = a_i \int_0^{2\pi} \cos^2(ix)\, dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\, dx$$

For $i = 0$,

$$\int_0^{2\pi} f(x)\, dx = \frac{a_0}{2} \cdot 2\pi$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx$$

Conditions: (Dirichilet condition)

1. Finite discontinuities
2. Finite number of extreme values

$\Rightarrow$ Piecewise regular

Exponential form:

$$f(x) = \frac{1}{2} \cdot e^{i0x} \cdot a_0 + \sum_{n=1}^{\infty} \cdot \frac{1}{2} a_n (e^{inx} + e^{-inx}) + \sum_{n=1}^{\infty} \frac{1}{2i} b_n (e^{inx} - e^{-inx})$$

$$= \frac{1}{2} e^{i0x} \cdot a_0 + \frac{1}{2} \sum (a_n - ib_n) e^{inx} + \frac{1}{2} \sum (a_n + ib_n) e^{-inx}$$

$$= \sum_{n=-\infty}^{+\infty} c_n e^{inx}$$

where

$$c_n = \frac{1}{2}(a_n - i b_n) \qquad c_{-n} = \frac{1}{2}(a_n + i b_n) \quad , \quad n > 0$$

$$c_0 = \frac{1}{2} a_0$$

Sturm - Liouville Theory

$$\mathcal{L} \psi(x) = \lambda \psi(x)$$

$$\mathcal{L}(x) = p_0(x) \frac{d^2}{dx^2} + p_1(x) \frac{d}{dx} + p_2(x)$$

If $\mathcal{L}$ is self-adjoint, then

$$p_0'(x) = p_1(x)$$

$$\Rightarrow \mathcal{L} = \frac{d}{dx}\left[p_0(x) \frac{d}{dx}\right] + p_2(x)$$

$$\mathcal{L} u = (p_0 u')' + p_2 u$$

Consider

$$\int_a^b v^* \mathcal{L} u \, dx = \int_a^b v^* (p_0 u')' \, dx + \int_a^b v^* p_2 u \, dx$$

$$= v^* p_0 u' \Big|_a^b + \int_a^b \left[-(v^*)' p_0 u' + v^* p_2 u \right] dx$$

$$= \left[v^* p_0 u' - (v^*)' p_0 u \right]_a^b + \int_a^b \left[[p_0 (v^*)']' u + v^* p_2 u \right] dx$$

$$= \left[v^* p_0 u' - (v^*)' p_0 u \right]_a^b + \int_a^b (\mathcal{L} v)^* u \, dx$$

If u, v are both eigenfunctions of $\mathcal{L}$, then

$$(\lambda_u - \lambda_v)\int_a^b v^* u \, dx = \left[p_0 \left(v^* u' - (v^*)' u \right) \right]_a^b$$

So, if $\lambda_u \neq \lambda_v$ and the boundary term vanishes,

$$\int_a^b v^* u \, dx = 0 \qquad \text{orthogonal} .$$

For our case.

$$-y''(x) = \lambda \cdot y(x).$$
$$y(0) = y(2\pi)$$
$$y'(0) = y'(2\pi)$$

eigenfunctions

① $\cos nx$, $\sin nx$

② e^{inx}

$$\langle \sin nx | \sin nx \rangle = \langle \cos nx | \cos nx \rangle = \pi$$
$$\langle e^{inx} | e^{inx} \rangle = 2\pi$$

So, $\varphi_n = \dfrac{\cos nx}{\sqrt{\pi}}$ $\qquad \varphi_{-n} = \dfrac{\sin nx}{\sqrt{\pi}}$ $\qquad \varphi_0 = \dfrac{1}{\sqrt{2\pi}}$

or $\varphi_n = \dfrac{e^{inx}}{\sqrt{2\pi}}$

Discontinous Functions.

Consider Fourier series for $|n| \leq r$

$$f_r(x) = \sum_{n=-r}^{r} C_n e^{inx} \quad , \quad C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt$$

So,

$$f_r(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \sum_{n=-r}^{r} e^{i(x-t)\cdot n} dt$$

Since

$$\sum_{n=-r}^{r} y^n = \frac{y^{-r} - y^{r+1}}{1-y} = \frac{y^{r+\frac{1}{2}} - y^{-(r+\frac{1}{2})}}{y^{\frac{1}{2}} - y^{-\frac{1}{2}}}$$

So,

$$f_r(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \cdot \frac{\sin[(r+\frac{1}{2})(x-t)]}{\sin[\frac{1}{2}(x-t)]} dt$$

$$D_r(u) := \frac{\sin[(r+\frac{1}{2})u]}{\sin \frac{1}{2}u} \qquad \text{Dirichilet kernel}$$

$$\Rightarrow f_r(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_r(x-t) dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) D_r(t) dt$$

and $D_r(u)$ is even, so

$$f_r(x) = \frac{1}{2\pi} \int_{0}^{\pi} (f(x-t) + f(x+t)) D_n(t) dt$$

Riemann Lemma:

If $f: (w_1, w_2) \to \mathbb{R}$ is absolutely integrable on (w_1, w_2), then

$$\lambda \to \infty, \lambda \in \mathbb{R} \Rightarrow \int_{w_1}^{w_2} f(x) e^{i\lambda x} dx \to 0$$

So.

$$f_r(x) = \frac{1}{2\pi} \int_0^{\delta} (f(x-t) + f(x+t)) \frac{\sin(r+\frac{1}{2})t}{\sin\frac{1}{2}t} dt + o(1), \quad r \to \infty$$

Dini Condition.

$$f: \mathring{U}(x) \to \mathbb{C}$$

If $f(x_-) = \lim_{t \to 0^+} f(x-t)$ and $f(x_+) = \lim_{t \to 0^+} f(x+t)$ exist.

Then

$$\int_{\to 0} \frac{(f(x-t) - f(x_-)) + (f(x+t) - f(x_+))}{t} dt = \sigma(x)$$

For our case.

$$f_r(x) - \frac{f(x_-) + f(x_+)}{2}$$

$$= \frac{1}{\pi} \int_0^{\pi} \frac{(f(x-t) - f(x_-)) + (f(x+t) - f(x_+))}{2\sin\frac{1}{2}t} \cdot \sin(r+\frac{1}{2})t \, dt$$

Since $2\sin\frac{1}{2}t \sim t$. By the above, equates 0.

So $\lim_{r \to +\infty} f_r(x) = \frac{f(x_-) + f(x_+)}{2}$

Square Wave.

$$f(x) = \begin{cases} \frac{h}{2} & 0 < x < \pi \\ -\frac{h}{2} & -\pi < x < 0 \end{cases}$$

$f(x)$ is odd.

$$\frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} f(x) \cdot \sin(nx) dx$$

$$= \frac{1}{\sqrt{\pi}} \cdot 2 \cdot \int_0^{\pi} \frac{h}{2} \cdot \sin(nx) \, d(nx) \cdot \frac{1}{n}$$

$$= \frac{h}{n\sqrt{\pi}} (-\cos nx) \Big|_0^{\pi}$$

$$= \begin{cases} \frac{2h}{n\sqrt{\pi}} & , n \text{ is odd} \\ 0 & , n \text{ is even} \end{cases}$$

So, $f(x) = \frac{2h}{\pi} \left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots \right)$

$$f_r(x) = \frac{h}{4\pi} \left[\int_0^{\pi} \frac{\sin[(r+\frac{1}{2})(x-t)]}{\sin \frac{1}{2}(x-t)} dt - \int_{-\pi}^{0} \frac{\sin[(r+\frac{1}{2})(x-t)]}{\sin \frac{1}{2}(x-t)} \right]$$

$x - t \to s$ $\qquad$ $x - t \to -s$

$$= \frac{h}{4\pi} \left[\int_{-\pi+x}^{x} \frac{\sin(r+\frac{1}{2})s}{\sin \frac{1}{2} s} ds - \int_{-\pi-x}^{-x} \frac{\sin(r+\frac{1}{2})s}{\sin \frac{1}{2} s} ds \right]$$

$$\Phi(t) := \int_0^{t} \frac{\sin(r+\frac{1}{2})s}{\sin \frac{1}{2} s} ds$$

So, $f_r(x) = \frac{h}{4\pi} \left[\Phi(x) - \Phi(-\pi+x) - \Phi(-x) + \Phi(-\pi-x) \right]$

$$= \frac{h}{4\pi} \left[(\Phi(x) - \Phi(-x)) - (\Phi(-\pi+x) - \Phi(-\pi-x)) \right]$$

So,

$$f_r(x) = \frac{h}{4\pi}\int_{-x}^{x} \frac{\sin(r+\frac{1}{2})s}{\sin\frac{1}{2}s}ds - \frac{h}{4\pi}\int_{-\pi-x}^{-\pi+x} \frac{\sin(r+\frac{1}{2})s}{\sin\frac{1}{2}s}ds$$

$\downarrow$

0

$r \to +\infty,\ x \to 0$

$$f_r(x) = \frac{h}{4\pi}\int_{-x}^{x} \frac{\sin(r+\frac{1}{2})s}{\sin\frac{1}{2}s}ds$$

$r+\frac{1}{2} \to p \qquad ps \to \xi$

$$f_r(x) = \frac{h}{2\pi}\int_{0}^{px} \frac{\sin\xi}{\sin(\frac{\xi}{2p})}\frac{d\xi}{p}$$

Calculation of Overshoot

$f_r(0) = 0$

$px \to \pi$, $f_r(x)$ reaches maximum

$$f_r(x_{max}) = \frac{h}{2\pi}\int_0^{\pi}\frac{\sin\xi\, d\xi}{p\cdot\sin(\frac{\xi}{2p})} \approx \frac{h}{\pi}\int_0^{\pi}\frac{\sin\xi}{\xi}d\xi$$

$$= \frac{h}{2}\cdot\frac{2}{\pi}\int_0^{\pi}\frac{\sin\xi}{\xi}d\xi$$

$$\int_0^{\pi}\frac{\sin\xi}{\xi} = \int_0^{+\infty}\frac{\sin\xi}{\xi}d\xi - \int_{\pi}^{+\infty}\frac{\sin\xi}{\xi}d\xi$$

$$= \frac{\pi}{2} - \left(\int_{\pi}^{3\pi} + \int_{3\pi}^{5\pi} + \int_{5\pi}^{7\pi} + \cdots\right)\frac{\sin\xi}{\xi}d\xi > \frac{\pi}{2}$$

So $\quad f_r(x_{max}) > \frac{\pi}{2} \quad , f_r(x_{max}) = \frac{2}{\pi}\int_0^{\pi}\frac{\sin\xi}{\xi}d\xi \cdot f(x_{max})$

$$\frac{2}{\pi}\int_0^{\pi}\frac{\sin\xi}{\xi}d\xi = 1.1789797\ldots$$