Chapter 20

Integral Transforms

Introduction

$$g(x) = \int_a^b f(t)\, K(x,t)\, dt$$

$$g(x) = \mathcal{L} f(t)$$

Linearity:

$$\int_a^b [f_1(t) + f_2(t)]\, K(x,t)\, dt = \int_a^b f_1(t)\, K(x,t)\, dt + \int_a^b f_2(t)\, K(x,t)\, dt$$

$$\int_a^b c f(t)\, K(x,t)\, dt = c \int_a^b f(t)\, K(x,t)$$

In order to be useful, there should exist $\mathcal{L}^{-1}$ and convenient to calculate.

$$\mathcal{L}^{-1} g(x) = f(t)$$

$$\begin{tikzcd}
\text{Original problem} \arrow[r, dashed] \arrow[d] & \text{Solution} \\
\text{Problem in transform space} \arrow[r] & \text{Solution in transform space} \arrow[u]
\end{tikzcd}$$

Some important transforms

Fourier transform

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)\, e^{i\omega t}\, dt$$

time $\rightarrow$ frequency

Laplace transform

$$F(s) = \int_{0}^{\infty} e^{-ts} f(t)\, dt$$

differential equations $\rightarrow$ algebraic equations

Hankel transform

$$g(\alpha) = \int_{0}^{+\infty} f(t) \cdot t\, J_n(\alpha t)\, dt$$

Mellin transform

$$g(\alpha) = \int_{0}^{\infty} f(t)\, t^{\alpha - 1}\, dt$$

example: $f(t) = e^{-t} \Rightarrow g(\alpha) = \Gamma(\alpha)$

Fourier Transform

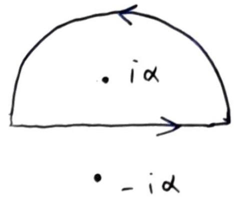
example 1. $f(t) = \delta(t)$

$$g(\omega) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{+\infty} \delta(t)\, e^{i\omega t}\, dt = \sqrt{\frac{1}{2\pi}}$$

example 2. $f(t) = \frac{2\alpha}{\alpha^2+t^2} \cdot \sqrt{\frac{1}{2\pi}}$

$$g(w) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{2\alpha e^{iwt}}{(t-i\alpha)(t+i\alpha)} dt$$

poles: $t = i\alpha$, $t = -i\alpha$



If $w > 0$,

$$g(w) = \frac{1}{2\pi} \cdot 2\pi i \cdot \frac{e^{-\alpha w}}{i}$$

$w < 0$

$$g(w) = \frac{1}{2\pi} \cdot -2\pi i \cdot \frac{e^{+\alpha w}}{-i}$$

$w = 0$

$$g(0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{2\alpha}{(t-i\alpha)(t+i\alpha)} = 1$$

$\therefore\ g(w) = e^{-\alpha|w|}$

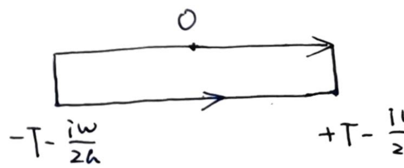
example 3: $f(t) = e^{-at^2}$, $a > 0$

$$g(w) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{+\infty} e^{-at^2} e^{iwt} dt$$

$$= \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{+\infty} e^{-a(t-\frac{iw}{2a})^2 - \frac{w^2}{4a}} d\left(t - \frac{iw}{2a}\right)$$

let $s = t - \frac{iw}{2a}$

$$\Rightarrow = \sqrt{\frac{1}{2\pi}} \cdot e^{-\frac{w^2}{4a}} \int_{-T-\frac{iw}{2a}}^{T-\frac{iw}{2a}} e^{-as^2} ds \quad (T \to +\infty)$$



$$= \sqrt{\frac{1}{2\pi}}\, e^{-\frac{w^2}{4a}} \cdot \int_{-\infty}^{+\infty} e^{-as^2} ds$$

$$= \sqrt{\frac{1}{2\pi}}\, e^{-\frac{w^2}{4a}} \cdot \sqrt{\frac{\pi}{a}}$$

$$= \sqrt{\frac{1}{2a}}\, e^{-\frac{w^2}{4a}}$$

Fourier Integral

$$f(x) = \lim_{n\to\infty} \int_{-\infty}^{+\infty} f(t)\delta(t-x)\,dt$$

$$= \lim_{n\to\infty} \int_{-\infty}^{+\infty} f(t) \cdot \frac{1}{2\pi}\left[\int_{-n}^{n} e^{iw(t-x)}\,dw\right] dt$$

$$= \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} dw \int_{-\infty}^{+\infty} dt \left[f(t)\, e^{iw(t-x)}\right]$$

$$= \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{+\infty} e^{-iwx}\,dw \underline{\int_{-\infty}^{+\infty} \sqrt{\frac{1}{2\pi}}\, f(t)\, e^{iwt}\,dt}$$

So.

$$f(t) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{+\infty} g(w)\, e^{-iwt}\,dt$$

For sine and cosine

$$g_c(w) = \sqrt{\frac{2}{\pi}} \int_{0}^{+\infty} f(t) \cos wt\,dt$$

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$$g_s(w) = \sqrt{\frac{2}{\pi}} \int_{0}^{+\infty} f(t) \sin wt\,dt$$

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example. Finite Wave Train

$$f(t) = \begin{cases} \sin w_0 t, & |t| < \frac{N\pi}{w_0} \\ 0, & |t| > \frac{N\pi}{w_0} \end{cases}$$

Since $f(t)$ is odd,

$$g_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\frac{N\pi}{\omega_0}} \sin\omega_0 t \sin\omega t \, dt$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\frac{N\pi}{\omega_0}} -\frac{1}{2}\left(\cos(\omega_0+\omega)t - \cos(\omega_0-\omega)t \right) dt$$

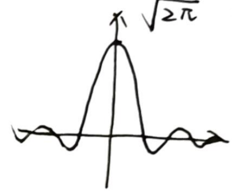
$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin[(\omega_0-\omega)(N\pi/\omega_0)]}{2(\omega-\omega_0)} - \frac{\sin[(\omega_0+\omega)(N\pi/\omega_0)]}{2(\omega+\omega_0)} \right]$$

For large ω_0 and $\omega \approx \omega_0$, vanishes.

Zeros:

$$(\omega_0-\omega)\cdot\frac{N\pi}{\omega_0} = k\pi \quad (k\in\mathbb{Z})$$

$\sqrt{\frac{1}{2\pi}}\frac{N\pi}{\omega_0}$



$$\frac{\Delta\omega}{\omega_0} = \frac{k}{N} \quad (k\in\mathbb{Z})$$

$$\Delta\omega = \frac{\omega_0}{N}$$ half width of the maximum.

Transform in 3D space

$$g(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int f(\vec{r})\, e^{i\vec{k}\cdot\vec{r}}\, d^3r$$

$$f(\vec{r}) = \frac{1}{(2\pi)^{3/2}} \int g(\vec{k})\, e^{-i\vec{k}\cdot\vec{r}}\, d^3k.$$

example. Yukawa potential $\frac{e^{-\alpha r}}{r}$

$$\left[\frac{e^{-\alpha r}}{r}\right]^T(k) = \frac{1}{(2\pi)^{3/2}} \int \frac{e^{-\alpha r}}{r}\cdot e^{i\vec{k}\cdot\vec{r}}\, d^3r$$

$$= \frac{4\pi}{(2\pi)^{3/2}} \int_0^{+\infty} r\,dr \int d\Omega_r \sum_{lm} i^l e^{-\alpha r} j_l(kr)\, Y_l^m(\Omega_k)^* \, Y_l^m(\Omega_r)$$

$l = m = 0$. or, vanish

$$Y_0^0 = \frac{1}{\sqrt{4\pi}}$$

$$\left[\frac{e^{-\alpha r}}{r}\right]^T (\vec{k}) = \frac{4\pi}{(2\pi)^{3/2}} \int_0^\infty r e^{-\alpha r} j_0(kr)\, dr$$

$$j_0(kr) = \frac{\sin kr}{r}$$

$$\Rightarrow \quad = \frac{4\pi}{(2\pi)^{3/2}} \int_0^\infty e^{-\alpha r} \sin kr\, dr$$

$$= \frac{1}{(2\pi)^{3/2}} \cdot \frac{4\pi}{k^2 + \alpha^2}$$

Coulomb potential

$$\left[\frac{1}{r}\right]^T (\vec{k}) = \frac{1}{(2\pi)^{3/2}} \cdot \frac{4\pi}{k^2}$$

Hydrogenic 1s orbital $\quad e^{-Zr}$

$$\left[e^{-Zr}\right]^T (\vec{k}) = -\frac{\partial}{\partial Z}\left[\frac{e^{-Zr}}{r}\right]^T (\vec{k}) = \frac{1}{(2\pi)^{3/2}} \frac{8\pi Z}{(k^2 + Z^2)^2}$$

Laplace Transforms.

Definition

$$f(s) = \mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\,dt$$

while $\int_0^\infty F(t)\,dt$ need not exist

$\mathcal{L}\{e^{t^2}\}$ ✗.

$\mathcal{L}\{t^n\}$ $\begin{cases} \checkmark \\ \times \quad n \le -1 \quad \text{diverge.} \end{cases}$

linearity:

$$\mathcal{L}\{aF(t) + bG(t)\} = a\mathcal{L}\{F(t)\} + b\mathcal{L}\{G(t)\}$$

- example

$$F(t) = \begin{cases} 1, & t > 0 \\ 0, & t \le 0 \end{cases}$$

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt = \frac{1}{s}, \quad s > 0$$

$$F(t) = e^{kt}, \quad t > 0$$

$$\mathcal{L}\{e^{kt}\} = \int_0^\infty e^{-st} e^{kt}\,dt = \frac{1}{s-k}, \quad s > k$$

$$\mathcal{L}\{\cosh\} = \frac{1}{2}\left\{\frac{1}{s-k} + \frac{1}{s+k}\right\} = \frac{s}{s^2-k^2}$$

$$\mathcal{L}\{\sinh\} = \frac{1}{2}\left\{\frac{1}{s-k} - \frac{1}{s+k}\right\} = \frac{k}{s^2-k^2}$$

$\cos kt = \cosh(ikt)$ $\qquad$ $\sin kt = \sinh(ikt)$

So. $k \to ik$

$$\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}$$

$$\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}$$ $\qquad$ $s \to 0$, $\mathcal{L}\{\sin kt\} \to \frac{1}{k}$

$$= \int_0^\infty e^{-st} \cdot \sin kt \, dt$$ $\qquad$ not exist.

$F(t) = t^n$

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st} \cdot t^n dt$$

$$= \frac{\Gamma(n+1)}{s^{n+1}}, \quad s>0, \ n>-1$$

$F(t) = \delta(t-t_0)$

$$\mathcal{L}\{\delta(t-t_0)\} = \int_0^\infty e^{-st}\delta(t-t_0)\,dt = e^{-st_0}$$

$$\mathcal{L}\{\delta(t)\} = 1$$ $\qquad$ impulse.

Inverse Transform.

$$\mathcal{L}^{-1}\{f(s)\} = F(t)$$

if $\int_0^{t_0} (F_1(t) - F_2(t))dt \equiv 0$, (null function)

then, $\mathcal{L}\{F_1(t)\} = \mathcal{L}\{F_2(t)\}$.

Lerch's theorem.

differs at isolated points.

1. Table .

2. Calculus of residues

⋮

example.

$$f(s) = \frac{k^2}{s(s^2+k^2)} = \frac{1}{s} - \frac{-s}{s^2+k^2}$$

$$\mathcal{L}^{-1}\{f(s)\} = 1-\cos(kt) \quad , t>0 .$$

example

$$F(t) = \int_0^{+\infty} \frac{\sin tx}{x}\, dx$$

$$\begin{aligned} f(s) &= \mathcal{L}\left\{ \int_0^{\infty} \frac{\sin tx}{x}\, dx \right\} \\ &= \int_0^{\infty} e^{-st} \int_0^{\infty} \frac{\sin tx}{x}\, dx\, dt \\ &= \int_0^{\infty} \frac{1}{x}\left[\int_0^{\infty} e^{-st} \sin tx\, dt \right] \\ &= \int_0^{\infty} \frac{dx}{s^2+x^2} \\ &= \frac{1}{s} \arctan\left(\frac{x}{s}\right)\Big|_0^{\infty} \\ &= \frac{\pi}{2s} . \end{aligned}$$

So. $F(t) = \frac{\pi}{2}$, $t>0$.

Properties of Laplace Transforms

$$\mathcal{L}\{F'(t)\} = \int_0^\infty e^{-st}\frac{dF(t)}{dt}\,dt$$

$$= e^{-st}F(t)\Big|_0^\infty + s\int_0^\infty e^{-st}F(t)\,dt$$

$$= s\mathcal{L}\{F(t)\} - F(0^+)$$

$$\mathcal{L}\{F''(t)\} = s^2\mathcal{L}\{F(t)\} - sF(0^+) - F'(0^+)$$

$$\mathcal{L}\{F^{(n)}(t)\} = s^n\mathcal{L}\{F(t)\} - s^{n-1}F(0^+) - \cdots - F^{(n-1)}(0^+)$$

example:

$$-k^2\sin kt = \frac{d^2}{dt^2}\sin kt$$

So. $\mathcal{L}\{-k^2\sin kt\} = \mathcal{L}\{\frac{d}{dt^2}\sin kt\}$

$$= s^2\mathcal{L}\{\sin kt\} - s\sin(0) - \frac{d}{dt}\sin kt\Big|_{t=0}$$

$$\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}$$

$$f'(s) = \int_0^\infty (-t)\,e^{-st}F(t)\,dt = \mathcal{L}\{-tF(t)\}$$

$$f^n(s) = \mathcal{L}\{(-t)^n F(t)\}$$

example. $\mathcal{L}\{e^{kt}\} = \int_0^\infty e^{-st}e^{kt}\,dt = \frac{1}{s-k}$, $s>k$

So, $\mathcal{L}\{te^{kt}\} = \frac{1}{(s-k)^2}$, $s>k$.

Integration of Transforms.

$$f(x) = \int_0^\infty e^{-xt} F(t)\, dt$$

$$\int_s^\infty f(x)\, dx = \int_s^\infty dx \int_0^\infty dt\, e^{-xt} F(t) = \int_0^\infty e^{-st} \frac{F(t)}{t}\, dt$$

$$= \mathcal{L}\left\{ \frac{F(t)}{t} \right\}$$

Inverse Laplace Transform.

$$F(t) \doteq \mathcal{L}^{-1}\{f(s)\}.$$

extract $e^{\beta t}$

$$F(t) = e^{\beta t} G(t).$$

$$G(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{iut}\, du \int_0^\infty G(v) e^{-iuv}\, dv$$

Insert it, we have

$$F(t) = \frac{e^{\beta t}}{2\pi} \int_{-\infty}^{\infty} e^{iut}\, du \int_0^\infty F(v) e^{-\beta v} \cdot e^{-iuv}\, dv$$

let $s = \beta + iu$

Then, $\displaystyle\int_0^\infty F(v) e^{-sv}\, dv = f(s)$ $\qquad du = \frac{ds}{i}$

$$F(t) = \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} e^{st} f(s)\, ds$$

Inverse transform. Bromwich integral.

Fourier-Mellin integral

$F(t) = \sum \left(\text{residues } Re(s) < \beta \right)$ analytic.