Chapter 21

Integral Equations

Introduction.

linear.

$$f(x) = \int_a^b K(x,t)\varphi(t)\,dt \qquad 1$$

Fredholm equation

$$\varphi(x) = f(x) + \lambda \int_a^b K(x,t)\varphi(t)\,dt \qquad 2$$

$$f(x) = \int_a^x K(x,t)\varphi(t)\,dt \qquad 1$$

Volterra equation

$$\varphi(x) = f(x) + \int_a^x K(x,t)\varphi(t)\,dt \qquad 2$$

Transformation:

Differential $\longrightarrow$ Integral

linear second-order ordinary differential equation.

$$y'' + A(x)y' + B(x)y = g(x)$$

initial conditions:

$$y(a) = y_0 \qquad y'(a) = y_0'$$

Integrate.

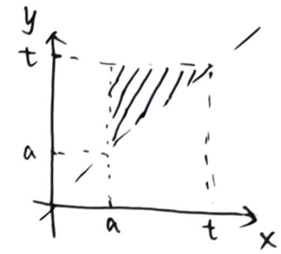
$$y'(x) = -\underline{\int_a^x A(t)y'(t)\,dt} - \int_a^x B(t)y(t)\,dt + \int_a^x g(t)\,dt + y_0'$$

$$= -A(x)y(x) - \int_a^x [B(t) - A(t)]y(t)\,dt + \cdots$$

Integrate again.

$$y(x) = -\int_a^x A(t)\,y(t)\,dt - \int_a^x du \int_a^u \left[B(t) - A'(t)\right] y(t)\,dt$$
$$+ \int_a^x du \int_a^u g(t)\,dt + \left(A(a)\,y_0 + y_0'\right)(x-a) + y_0$$

To make it neater,



$$\int_a^x du \int_a^u f(t)\,dt \;\rightarrow\; \int_a^t dy \int_a^y f(x)\,dx$$

$$\|$$

$$\int_a^x f(t)\,dt \int_t^x du \;\leftarrow\; \int_a^t dx \int_x^t f(x)\,dy$$

$$\|$$

$$\int_a^x f(t)\cdot(x-t)\,dt$$

So,
$$y(x) = -\int_a^x \underbrace{\left(A(t) + (x-t)\left[B(t) - A'(t)\right]\right)}_{\rightarrow\, K(x,t)} y(t)\,dt$$
$$+ \underbrace{\int_a^x (x-t)\,g(t)\,dt + \left[A(a)\,y_0 + y_0'\right](x-a) + y_0}_{\rightarrow\, f(x)}$$

$$\Rightarrow \quad y(x) = f(x) + \int_a^x K(x,t)\,y(t)\,dt$$

Volterra equation of the second kind.

example: linear oscillator

$$y'' + w^2 y = 0$$

$$y(0) = 0, \quad y'(0) = 1$$

$$\Rightarrow \quad A(x) = 0 \qquad B(x) = w^2 \qquad g(x) = 0$$

So

$$y(x) = x + w^2 \int_0^x (t-x)\,y(t)\,dt$$

.

It's easy to check that

$$y(x) = \frac{1}{w} \sin(wx)$$

is its solution.

If the boudary condition becomes

$$y(0) = 0 \qquad y(b) = 0$$

We don't know $y'(0)$, so, Integrate.

$$y' = -w^2 \int_0^x y\,dx + y'(0)$$

Integrate again.

$$y = -w^2 \int_0^x (x-t)\,y(t)\,dt + x y'(0)$$

use $y(b) = 0$,

$$w \int_0^b (b-t)\,y(t)\,dt = b y'(0)$$

Substitute back.

$$y = -w^2 \int_0^x (x-t)\,y(t)\,dt + x \cdot \frac{w}{b} \int_0^b (b-t)\,y(t)\,dt$$

$$\left(\int_0^x + \int_x^b\right)$$

Since

$$(x-t) = \frac{x}{b}(b-t) - \frac{t}{b}(b-x)$$

we find

$$y(x) = w^2 \int_0^x \frac{t}{b}(b-x)\,y(t)\,dt + w^2 \int_x^b \frac{x}{b}(b-t)\,y(t)\,dt$$

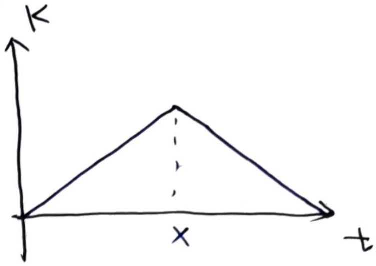
So, the kernel

$$K(x,t) = \begin{cases} \frac{t}{b}(b-x) & , t < x \\ \frac{x}{b}(b-t) & , t > x \end{cases}$$

$$y(x) = w^2 \int_0^b K(x,t)\,y(t)\,dt$$

The kernel:

1. symmetric. $K(x,t) = K(t,x)$
2. continuous
3. $\frac{\partial K}{\partial t}$ is discontinuous



Green's function.

Initial condition $\longrightarrow$ Volterra

Boundary condition $\longrightarrow$ Fredholm.

differential $\rightleftarrows$ integral,

Some special methods.

1. Integral transform.

$$f(x) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{+\infty} e^{ixt} \varphi(t)\, dt$$ Fourier

Solution $$\varphi(x) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{+\infty} e^{-ixt} f(t)\, dt$$

$$f(x) = \int_{0}^{+\infty} e^{xt} \varphi(t)\, dt$$ Laplace

$$\varphi(x) = \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} e^{xt} f(t)\, dt$$

$$f(x) = \int_{0}^{\infty} t^{x-1} \varphi(t)\, dt$$ Mellin

$$\varphi(x) = \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} x^{-t} f(t)\, dt$$

$$f(x) = \int_0^{\infty} t\varphi(t)\, J_\nu(xt)\, dt$$

Hankel

$$\varphi(x) = \int_0^{\infty} t f(t)\, J_\nu(xt)\, dt$$

Fourier Convolution theorem

$$(f * g)(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(y) f(x-y)\, dy$$

3D:

$$(f * g)(\vec{r}) := \frac{1}{(2\pi)^{3/2}} \int g(\vec{r}\,')\, f(\vec{r} - \vec{r}\,')\, d^3 r'$$

Faltung : folding

example: $f(y) = e^{-y}$, $f(x-y) = e^{-(x-y)}$

$$(f * g)^T(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx\, e^{itx} \left[\int_{-\infty}^{+\infty} dy\, g(y) f(x-y) \right]$$

$$= \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dy\, g(y) e^{ity} \right] \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx\, f(x-y)\, e^{it(x-y)} \right]$$

$$= G(t)\, F(t)$$

Inverse:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} G(t) F(t) e^{-ixt} dt = (f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(y) f(x-y)\, dy$$

Laplace Convolution Theorem.

$$f_1(s) = \mathcal{L}\{F_1(t)\} \qquad f_2(s) = \mathcal{L}\{F_2(t)\}$$

$$f_1(s) f_2(s) = \int_0^\infty e^{-sx} F_1(x)\,dx \int_0^\infty e^{-sy} F_2(y)\,dy$$

let $t = x + y$

$$f_1(s) f_2(s) = \int_0^\infty e^{-st}\,dt \int_0^t F_1(t-y) F_2(y)\,dy$$

$$= \mathcal{L}\left\{ \int_0^t F_1(t-y) F_2(y)\,dy \right\}$$

$$= \mathcal{L}\{F_1 * F_2\}$$

$$F_1 * F_2 := \int_0^t F_1(t-z) F_2(z)\,dz$$

Inverse:

$$\mathcal{L}^{-1}\{f_1(s) f_2(s)\} = F_1 * F_2$$

Example. Fourier Transform Solution.

$$f(x) = \int_{-\infty}^{+\infty} K(x-t)\varphi(t)\,dt$$

from fourier convolution theorem

$$f(x) = \int_{-\infty}^{+\infty} K(w)\Phi(w) e^{-iwx}\,dw$$

taking fourier transform.

$$K(w)\Phi(w) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x) e^{iwx}\,dx = \frac{F(w)}{\sqrt{2\pi}}$$

$$\Phi(w) = \frac{1}{\sqrt{2\pi}} \frac{F(w)}{K(w)}$$

$$\varphi(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{F(w)}{K(w)} e^{-iwx}\,dw$$

- example: Generalized Abel Equation

$$f(x) = \int_0^x \frac{\varphi(t)}{(x-t)^\alpha}\, dt \quad , \; 0 < \alpha < 1$$

Taking the Laplace transform.

$$\mathcal{L}\{f(x)\} = \mathcal{L}\left\{\int_0^x \frac{\varphi(t)}{(x-t)^\alpha}\, dt\right\} = \mathcal{L}\{x^{-\alpha}\}\, \mathcal{L}\{\varphi(x)\}$$

$$\mathcal{L}\{\varphi(x)\} = \frac{s^{1-\alpha}}{\Gamma(1-\alpha)} \cdot \mathcal{L}\{f(x)\}$$

$$\frac{1}{s}\mathcal{L}\{\varphi(x)\} = \frac{s^{-\alpha}}{\Gamma(1-\alpha)}\, \mathcal{L}\{f(x)\} = \frac{\mathcal{L}\{x^{\alpha-1}\}\, \mathcal{L}\{f(x)\}}{\Gamma(\alpha)\, \Gamma(1-\alpha)}$$

$$\downarrow$$

$$\mathcal{L}\left\{\int_0^x \varphi(t)\, dt\right\} = \frac{\sin \pi\alpha}{\pi}\, \mathcal{L}\left\{\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}\, dt\right\}$$

$$\int_0^x \varphi(t)\, dt = \frac{\sin \pi\alpha}{\pi} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}\, dt$$

$$\varphi(t) = \frac{\sin \pi a}{\pi} \frac{d}{dx} \int_0^x \frac{f(t)}{(x-t)}\, dt$$

Generating Function Method

$$f(x) = \int_{-1}^{+1} \frac{\varphi(t)}{\sqrt{1-2xt+x^2}}\, dt \quad , \; -1 \le x \le 1$$

orthonormal

Legendre polynomials.

$$\frac{1}{\sqrt{1-2xt+x^2}} = \sum_{n=0}^{\infty} P_n(t)\, x^n$$

$$\varphi(t) = \sum_{m=0}^{\infty} a_m P_m(t)$$

$$\therefore f(x) = \sum_{mn} a_m x^n \int_{-1}^{1} P_n(t)\, P_m(t)\, dt$$

$$= \sum_{mn} a_m x^n \frac{2\delta_{mn}}{2n+1} = \sum_{n=0}^{\infty} \frac{2a_n}{2n+1} x^n$$

Maclaurin expansion for $f(x)$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}$$

So, equate them.

$$\frac{f^{(n)}(0)}{n!} = \frac{2a_n}{2n+1}$$

$$\varphi(t) = \sum_{m=0}^{\infty} a_m P_m(t)$$

$$= \sum_{n=0}^{\infty} \frac{2n+1}{2} \frac{f^{(n)}(0)}{n!} P_n(t)$$

Separable Kernel

$$K(x,t) = \sum_{j=1}^{n} M_j(x) N_j(t) \qquad , n < +\infty$$

degenerate

Polynomials and some elementary transdental function

as.

$$K(x,t) = \cos(t-x)$$

$$= \cos t \cos x + \sin t \sin x$$

For a fredholm equation. (2)

$$\varphi(x) = f(x) + \lambda \sum_{j=1}^{n} M_j(x) \underbrace{\int_a^b N_j(t) \varphi(t)\, dt}_{C_j}$$

So,

$$\varphi(x) = f(x) + \lambda \sum_{j=1}^{n} C_j M_j(x)$$

multiply $N_i(x)$ and integrate

$$\underbrace{\int_a^b \varphi(x) N_i(x)\, dx}_{c_i} = \underbrace{\int_a^b f(x) N_i(x)\, dx}_{b_i} + \lambda \sum_{j=1}^{n} c_j \underbrace{\int_a^b M_j N_i(x)\, dx}_{a_{ij}}$$

$$c_i = b_i + \lambda \sum_j a_{ij} c_j$$

$$\vec{c} = \vec{b} + \lambda \mathbb{A} \vec{c}$$

$$\vec{b} = (\mathbb{1} - \lambda \mathbb{A}) \vec{c} \qquad \vec{c} = (\mathbb{1} - \lambda \mathbb{A})^{-1} \vec{b}$$

if $f(x) = 0$, then $\vec{b} = \vec{0}$

$$|\mathbb{1} - \lambda \mathbb{A}| = 0$$

example

$$\varphi(x) = \lambda \int_{-1}^{1} (t + x) \varphi(t)\, dt$$

$$t + x = 1 \cdot t + x \cdot 1 = M_1(x) N_1(t) + M_2(x) N_2(t)$$

$$a_{ij} = \int_{-1}^{1} N_i(x) M_j(x)\, dx$$

So,

$$\begin{vmatrix} 1 & -\frac{2\lambda}{3} \\ -2\lambda & 1 \end{vmatrix} = 0$$

$$\lambda = \pm \frac{\sqrt{3}}{2}$$

$$\vec{b} = (\mathbb{1} - \lambda \mathbb{A}) \vec{c} = \vec{0} \Rightarrow c_1 \mp \frac{c_2}{\sqrt{3}} = 0$$

if let $c_1 = 1$

$$\varphi_1(x) = \frac{\sqrt{3}}{2}(1 + \sqrt{3} x) \qquad \lambda = \frac{\sqrt{3}}{2}$$

$$\varphi_2(x) = -\frac{\sqrt{3}}{2}(1 - \sqrt{3} x) \qquad \lambda = -\frac{\sqrt{3}}{2}$$

21.3. Neumann Series

$$\varphi(x) = f(x) + \lambda \int_a^b K(x,t)\varphi(t)\,dt \quad , \; f(x) \neq 0$$

Suppose

$$\varphi_0(x) \approx f(x)$$

$$\varphi_1(x) = f(x) + \lambda \int_a^b K(x,t) f(t)\,dt$$

$$\varphi_2(x) = f(x) + \lambda \int_a^b K(x,t_1) f(t_1)\,dt_1 + \lambda^2 \int_a^b \int_a^b K(x,t_1) K(t_1,t_2) f(t_2)\,dt_2\,dt_1$$

$$\varphi_n(x) = \sum_{i=0}^{n} \lambda^i u_i(x)$$

where

$$u_0(x) = f(x)$$

$$u_1(x) = \int_a^b K(x,t_1) f(t_1)\,dt_1$$

$$\vdots$$

$$u_n(x) = \underbrace{\int_a^b \int_a^b \cdots \int_a^b}_{n} \left[\prod_{i=1}^{n} K_i\right] f(t_n)\,dt_n \cdots dt_1$$

We expect

$$\varphi(x) = \lim_{n \to \infty} \varphi_n(x) = \lim_{n \to +\infty} \sum_{i=0}^{n} \lambda^i u_i(x)$$

$$|\lambda^n u_n(x)| \leq |\lambda^n| \cdot |f|_{max} \cdot |K|_{max}^n \cdot |b-a|^n$$

So, if $|\lambda||K|_{max}|b-a| < 1$, converge.

operator form

$$\varphi = \lambda \hat{K} \varphi + f$$

$$\hat{K} = \int_a^b K(x,t)\,[\;]\,dt$$

$$\Rightarrow \quad \varphi = (\mathbb{1} - \lambda \hat{K})^{-1} f$$

Binomial expansion

example.

$$\varphi(x) = x + \frac{1}{2}\int_{-1}^{1} (t-x)\varphi(t)\,dt$$

$$\varphi_0(x) = x$$

Then

$$\varphi_1(x) = x + \frac{1}{2}\int_{-1}^{1} (t-x)\,t\,dt = x + \frac{1}{3}$$

$$\varphi_2(x) = x + \frac{1}{3} - \frac{x}{3}$$

$$\varphi_3(x) = x + \frac{1}{3} - \frac{x}{3} - \frac{1}{3^2}$$

$$\vdots$$

$$\varphi(x) = \frac{3}{4}x + \frac{1}{4}.$$