Chapter 22

Calculus of Variations

Euler Equation

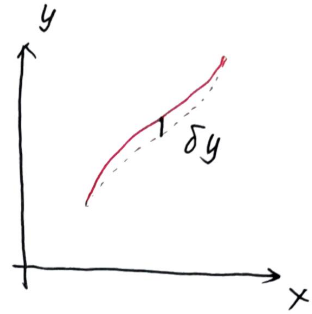
$$J[y] = \int_{x_1}^{x_2} f\left(y(x), \frac{dy(x)}{dx}, x\right) dx$$

We want to find y, s.t. J is stationary.

$$\delta J = \delta \int_{x_1}^{x_2} f(y, y_x, x)\, dx$$

example. chain in graritional field

brachistochrone problem.



$$\alpha \eta(x) := \delta y$$

$$\eta(x_1) = \eta(x_2) = 0$$

$$y(x, \alpha) = \underbrace{y(x, 0)}_{\text{minimize}} + \alpha \eta(x)$$

Then

$$J(\alpha) = \int_{x_1}^{x_2} f\left(y(x, \alpha), y_x(x, \alpha), x\right) dx$$

$$\left.\frac{\partial J(\alpha)}{\partial \alpha}\right|_{\alpha=0} = 0$$

$$\frac{\partial J(\alpha)}{\partial \alpha} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y_x} \cdot \frac{\partial y_x}{\partial \alpha} \right] dx = 0$$

and

$$\frac{\partial y}{\partial \alpha} = \eta(x) \quad , \quad \frac{\partial y_x}{\partial \alpha} = \frac{d\eta(x)}{dx}$$

So,

$$\frac{\partial J(\alpha)}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} \eta(x) + \underline{\frac{\partial f}{\partial y_x} \cdot \frac{d\eta(x)}{dx}} \right) dx = 0$$

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y_x} \frac{d\eta(x)}{dx} dx = \cancel{\frac{\partial f}{\partial y_x} \eta(x) \Big|_{x_1}^{x_2}} - \int_{x_1}^{x_2} \eta(x) \frac{d}{dx} \frac{\partial f}{\partial y_x} dx$$

So.

$$\frac{\partial J(\alpha)}{\partial \alpha} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y_x} \right] \eta(x)\, dx = 0 .$$

$$\Downarrow$$

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \cdot \frac{\partial f}{\partial y_x} = 0$$

Euler - Lagrange function

example.

$$J = \int_{x_1 y_1}^{x_2 y_2} ds = \int_{x_1}^{x_2} \sqrt{1 + y_x^2}\, dx$$

$$\text{So, } f(y, y_x, x) = \sqrt{1 + y_x^2}$$

Then,

$$\frac{d}{dx} \cdot \left(\frac{1}{\sqrt{1 + y_x^2}} \right) = 0$$

So, it's a constant,

$$\frac{1}{\sqrt{1+y_x^2}} = C$$

$$y_x = a.$$

$$y = kx + m$$

example. Optical Path Near A Black Hole.

Suppose $v(y) = \frac{y}{b}$

$y = 0, \ v = 0 \longrightarrow$ event horizon

$$\Delta t = \int dt = \int \frac{ds}{v} = \int \frac{b}{y} \cdot \sqrt{dx^2 + dy^2} \rightarrow \text{minimum}.$$

$$= b \int \frac{\sqrt{x_y^2 + 1}}{y} \, dy$$

So, $f(x, x_y, y) = \frac{\sqrt{x_y^2 + 1}}{y}$

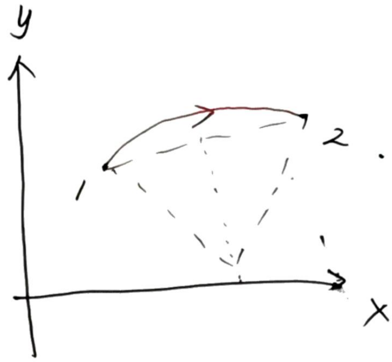
Substitute into E-L equation,

$$\frac{\partial f}{\partial x} - \frac{d}{dy} \frac{\partial f}{\partial x_y} = -\frac{d}{dy}\left(\frac{x_y}{y\sqrt{1+x_y^2}}\right) = 0.$$

So, $\frac{x_y}{y\sqrt{1+x_y^2}} = C_1 \Rightarrow \frac{dx}{dy} = \frac{C_1 y}{\sqrt{1 - C_1^2 y^2}}$

$$\int^{x} dx = \int^{y} \frac{C_1 y\, dy}{\sqrt{1 - C_1^2 y^2}}$$

$$x + C_2 = \frac{\sqrt{1 - C_1^2 y^2}}{C_1}$$



or, $$(x + C_2)^2 + y^2 = \frac{1}{C_1^2}$$

Arc.

desert.

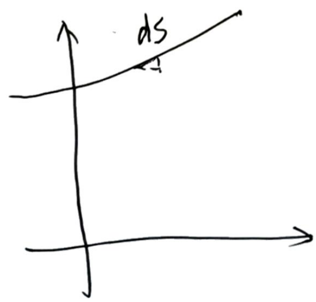
Alternate of E-L Equation.

$$\frac{\partial f}{\partial x} - \frac{d}{dx}\left(f - y_x \frac{\partial f}{\partial y_x}\right) = 0$$

expand.

if $f(y, y_x)$, then, $f - y_x \frac{\partial f}{\partial y_x} = \text{Const}$.

example: Soap Film.



$$dA = 2\pi y\, ds = 2\pi y \cdot \sqrt{1 + y_x^2}\, dx$$

So, $$f(y, y_x, x) = y \cdot \sqrt{(1 + y_x^2)}$$

$$y\sqrt{1 + y_x^2} - \frac{y_x \cdot y_x \cdot y}{\sqrt{1 + y_x^2}} = C_1$$

$$\frac{y}{\sqrt{1 + y_x^2}} = C_1$$

Squaring.

$$\frac{y^2}{1+y_x^2} = C_1^2$$

$$\frac{dy}{dx} = \frac{1}{C_1}\cdot\sqrt{y^2 - C_1^2}$$

$$x = C_1 \operatorname{arcosh}\frac{y}{C_1} + C_2$$

$$y = C_1 \cosh\left(\frac{x - C_2}{C_1}\right)$$

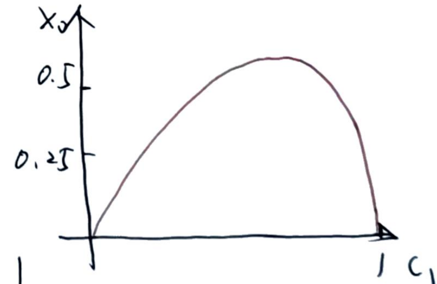
catenoid

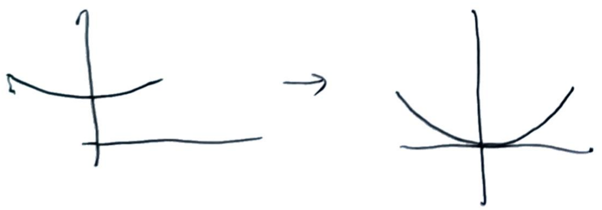
Consider $(-x_0, 1)$ $(x_0, 1)$. $\Rightarrow C_2 = 0$.

$$y = C_1 \cosh\left(\frac{x}{C_1}\right)$$

$$y(x_0) = C_1 \cosh\left(\frac{x_0}{C_1}\right) = 1$$

if $x_0 = 1$, no real solution.



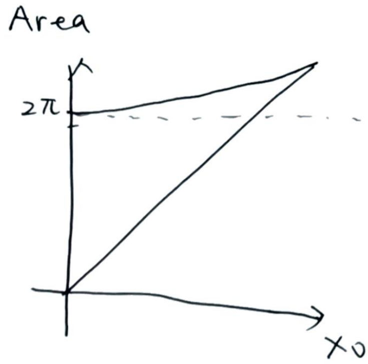


$$A = 4\pi\int_0^{y_0} y\sqrt{1+y_x^2}\,dx = \frac{4\pi}{C_1}\int_0^{x_0} y^2\,dx$$

$$= 4\pi C_1 \int_0^{x_0}\left(\cosh\frac{x}{C_1}\right)^2 dx$$

$$= \pi C_1^2\left[\sinh\left(\frac{2x_0}{C_1}\right) + \frac{2x_0}{C_1}\right]$$

Goldschmidt discontinuous solution $\longrightarrow$



More General Variations.

$$J = \int_{x_1}^{x_2} f\left(u_1(x), u_2(x) \cdots u_n(x), u_{1x}(x) \cdots u_n(x), x\right) dx$$

$$u_i(x, \alpha) = u_i(x, 0) + \alpha \eta_i(x)$$

$$\int_{x_1}^{x_2} \sum_i \left[\frac{\partial f}{\partial u_i} \eta_i + \frac{\partial f}{\partial u_{ix}} \eta_{ix}\right] dx = 0$$

$$\Rightarrow \int_{x_1}^{x_2} \sum_i \left[\frac{\partial f}{\partial u_i} - \frac{d}{dx}\frac{\partial f}{\partial u_{ix}}\right] \eta_i \, dx = 0$$

So,
$$\frac{\partial f}{\partial u_i} - \frac{d}{dx}\frac{\partial f}{\partial u_{ix}} = 0 \quad , i = 1, 2 \ldots n.$$

Hamilton's Principle.

$$\mathcal{L} := T - V$$

$x \rightarrow t$

$y_i \rightarrow x_i(t)$

$y_{ix} \rightarrow \dot{x}_i(t)$

$$\delta \int_{t_1}^{t_2} \mathcal{L}(x_1 \cdots \dot{x}_1 \cdots t)\, dt = 0$$

$\Rightarrow$ By E-L equation.

$$\frac{d}{dt} \cdot \frac{\partial \mathcal{L}}{\partial \dot{x}_i} - \frac{\partial \mathcal{L}}{\partial x_i} = 0$$

Lagrangian motion of motion.

generalized coordinates.

example. moving particle. in cartesian.

$$\mathcal{L} = \frac{1}{2} m \dot{x}^2 - V$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} \qquad \frac{\partial \mathcal{L}}{\partial x} = -\frac{dV}{dx} = F(x).$$

$$\frac{d}{dt}(m\dot{x}) = F(x)$$

in polar coordinates.

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} m (\dot{\rho}^2 + \rho^2 \dot{\varphi}^2)$$

$$V = 0.$$

So. $$\frac{d}{dt}(m\dot{\rho}) = m\rho\dot{\varphi}^2$$

$$\frac{d}{dt}(m\rho^2\dot{\varphi}) = 0$$

Hamilton's Equations

canonical momentum

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

So,

$$d\mathcal{L} = \sum_i \left(\frac{\partial \mathcal{L}}{\partial q_i} dq_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} d\dot{q}_i \right) + \frac{\partial \mathcal{L}}{\partial t} dt$$

$$d\left(\mathcal{L} - \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\dot{q}_i\right) = \sum_i \left(\frac{\partial \mathcal{L}}{\partial q_i} dq_i - \dot{q}_i \, d\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)\right) + \frac{\partial \mathcal{L}}{\partial t} dt$$

$$d\Bigg(\underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\dot{q}_i - \mathcal{L}}_{\shortparallel \atop H}\Bigg) = \sum_i \left(\dot{q}_i \, dp_i - \dot{p}_i \, dq_i \right) - \frac{\partial \mathcal{L}}{\partial t} dt$$

and

$$dH = \sum_i \left(\frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i \right) + \frac{\partial H}{\partial t} dt$$

So,

energy

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \qquad \frac{\partial H}{\partial q_i} = -\dot{p}_i \qquad \frac{\partial H}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}$$

Geodesics

$\rightarrow$ local minimum

$$ds^2 = \underset{\sim}{g_{ij}}\, dq^i dq^j$$

metric tensor. $g_{ij} = g_{ji}$

in Euclidean space,

$$ds^2 = dx^2 + dy^2 + dz^2.$$

$$J = \int_A^B \frac{ds}{du}\, du = \int_A^B \frac{\sqrt{g_{ij}\, dq^i\, dq_j}}{du}\, du$$

$$= \int_A^B \sqrt{g_{ij} \frac{dq^i}{du} \frac{dq^j}{du}}\, du$$

$$= \int_A^B \sqrt{g_{ij}\, \dot{q}^i\, \dot{q}^j}\, du.$$

Local minimum.

$$\mathcal{L} = \frac{m}{2} g_{ij}\, \dot{q}^i \dot{q}^j$$

$u \rightarrow \tau$ / affine trans~

$a\tau + b$

$$\delta \int_A^B g_{ij}\, \dot{q}^i\, \dot{q}^j\, du = 0$$

$$\Rightarrow \frac{\partial g_{ij}\, \dot{q}^i\, \dot{q}^j}{\partial q^k} - \frac{d}{du} \frac{\partial g_{ij}\, \dot{q}^i\, \dot{q}^j}{\partial \dot{q}^k} = 0$$

$$\dot{q}^i \dot{q}^j \frac{\partial g_{ij}}{\partial q^k} - \frac{d}{du} g_{ij} \left(\delta_{ik}\, \dot{q}^j + \dot{q}^i\, \delta_{jk} \right) = 0$$

$$\dot{q}^i \dot{q}^j \frac{\partial g_{ij}}{\partial q^k} - \frac{d}{du} \left(g_{kj}\, \dot{q}^j + g_{ik}\, \dot{q}^i \right) = 0$$

expand.

$$\dot{q}^i \dot{q}^j \frac{\partial g_{ij}}{\partial q^k} - \left(\frac{\partial g_{kj}}{\partial q^i} \dot{q}^j \dot{q}^i + \underbrace{g_{kj} \ddot{q}^j}_{g_{ik} \ddot{q}^i} + \frac{\partial g_{ik}}{\partial q^j} \dot{q}^i \dot{q}^j + g_{ik} \ddot{q}^i \right) = 0$$

$$\frac{1}{2} \dot{q}^i \dot{q}^j \left(\frac{\partial g_{ij}}{\partial q^k} - \frac{\partial g_{kj}}{\partial q^i} - \frac{\partial g_{ik}}{\partial q^j} \right) - g_{ik} \ddot{q}^i = 0$$

multiply $-g^{kl}$, $g^{kl} g_{ik} = \delta^l_i$

$$g^{kl} g_{ik} \ddot{q}^i - \frac{dq^i}{du} \frac{dq^j}{du} \cdot \frac{1}{2} g^{kl} \left[\frac{\partial g_{kj}}{\partial q^i} + \frac{\partial g_{ik}}{\partial q^j} - \frac{\partial g_{ij}}{\partial q^k} \right] = 0$$

use Christoffel symbol, we have

$$\frac{d^2 q^l}{du^2} + \frac{dq^i}{du} \frac{dq^j}{du} \Gamma^l_{ij} = 0$$