9.22:

We want to find a most general form of P_1 and P_2 for this point transformation,

$$\begin{bmatrix} Q_1 \\ Q_2 \\ P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} q_1^2 \\ q_1 + q_2 \\ P_1\,(q,p) \\ P_2\,(q,p) \end{bmatrix}.$$

We can define,

$$F_2(q_1, P_1, q_2, P_2) = q_1^2 P_1 + (q_1 + q_2)P_2 + g(q_1, q_2).$$

Since this transformation is canonical, we have,

$$\begin{aligned} p_1 =& \frac{\partial F_2}{\partial q_1} = 2q_1 P_1 + P_2 + \frac{\partial g}{\partial q_1}, \\ p_2 =& P_2 + \frac{\partial g}{\partial q_2}. \end{aligned}$$

Inverting them, we find,

$$\begin{aligned} P_1 =& \frac{p_1 - p_2}{2q_1} + \frac{1}{2q_1}(\frac{\partial g}{\partial q_2} - \frac{\partial g}{\partial q_1}), \\ P_2 =& p_2 - \frac{\partial g}{\partial q_2}. \end{aligned}$$

We want to choose a function $g(q_1, q_2)$ such that the Hamiltonian is cyclic in Q_1 and Q_2,

$$H'(P_1, P_2) = H = (\frac{p_1 - p_2}{2q_1})^2 + p_2 + (q_1 + q_2)^2$$

$$\implies H' = [P_1 - \frac{1}{2q_1}(\frac{\partial g}{\partial q_2} - \frac{\partial g}{\partial q_1})]^2 + P_2 + \frac{\partial g}{\partial q_2} + (2q_1 + q_2)^2.$$

We can choose,

$$g(q_1, q_2) = -\frac{1}{3}(q_1 + q_2)^2.$$

Then the new Hamiltonian is,

$$H' = P_1^2 + P_2.$$

Therefore,

$$\begin{aligned} \dot{Q}_1 =& 2P_1 \implies Q_1 = 2P_1 t + Q_{10}, \\ \dot{Q}_2 =& 1 \implies Q_2 = t + Q_{20}, \\ \dot{P}_1 =& 0 \implies P_1 = Const, \\ \dot{P}_2 =& 0 \implies P_2 = Const. \end{aligned}$$

And,

$$\begin{aligned} q_1 =& \sqrt{q_{10}^2 + 2P_1 t}, \\ q_2 =& q_{10} + q_{20} + t - \sqrt{q_{10}^2 + 2P_1 t}, \\ p_1 =& \frac{\sqrt{q_{10}^2 + 2P_1 t}}{q_{10}}(p_{10} - p_{20}) + p_{20} - 2(q_{10} + q_{20}), \\ p_2 =& p_{20} - 2(q_{10} + q_{20})t - t^2. \end{aligned}$$

9.25:

(a) By Hamilton's equation of motion, we have,

$$\dot{q} = \frac{\partial H}{\partial p} = pq^4,$$
$$\dot{p} = -\frac{\partial H}{\partial q} = -2p^2q^4 + \frac{1}{q^3}.$$

Since the Hamiltonian doesn't depend on t explicitly, energy is conserved.

$$E = \frac{\dot{q}^2}{2q^4} + \frac{1}{2q^2}.$$

With initial conditions $\dot{q}(0) = 0, q(0) = q_0$, we find that,

$$\frac{1}{2q_0^2} = \frac{\dot{q}^2}{2q^4} + \frac{1}{2q^2}$$
$$\implies q = \frac{q_0}{\cos t}$$

(b) What we need to find is,

$$H'(Q, P) = H(q, p) = \frac{1}{2}P^2 + \frac{1}{2}Q^2 = \frac{1}{2}p^2q^4 + \frac{1}{2q^2}.$$

It's obvious to find that,

$$Q = \frac{1}{q},$$
$$P = -pq^2.$$

We can verify that the transformation is canonical, that,

$$[Q, P]_{qp} = 1.$$

For the Hamiltonian, we have,

$$\ddot{Q} + Q = 0 \quad \text{and} \quad P = \dot{Q}.$$

Solve the equations,

$$Q = \frac{\cos t}{q_0} = \frac{1}{q},$$
$$P = \frac{-\sin t}{q_0} = -pq^2,$$

which are agreed with the results in (a).

9.29:

In Kepler's problem, we can express

$$a = -\frac{k}{2E},$$

$$e = \sqrt{1 + \frac{2El^2}{mk^2}},$$

$$\omega = \sqrt{\frac{k}{ma^3}} = \sqrt{-\frac{8E^3}{mk^2}},$$

$$\psi = 2\arctan\left[\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right]$$

$$T = \frac{2\pi - \psi + e\sin\psi}{\omega}$$

where $E = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{k}{r}, l = mr^2\dot{\theta}$.

The Poisson brackets are,

$$[a, e] = -\frac{2mr^5\dot{\theta}^3}{k\left(r^2\dot{\theta}^2 + \dot{\theta}^2\right)\sqrt{\frac{k^2+mr^6\dot{\theta}^4+mr^4\dot{\theta}^2\dot{\theta}^2}{k^2}}}$$

$$[a, \psi] = \frac{2kmr^4\dot{\theta}^2\left(\dot{\theta}\sqrt{\frac{k^2+mr^6\dot{\theta}^4+mr^4\dot{\theta}^2\dot{\theta}^2}{k^2}} + 2r\dot{\theta}\sin\theta\right)}{\left(r^2\dot{\theta}^2 + \dot{\theta}^2\right)\sqrt{\frac{k^2+mr^6\dot{\theta}^4+mr^4\dot{\theta}^2\dot{\theta}^2}{k^2}}\left(2k^2\cos\theta\sqrt{\frac{k^2+mr^6\dot{\theta}^4+mr^4\dot{\theta}^2\dot{\theta}^2}{k^2}} + 2k^2 + mr^4\dot{\theta}^2\left(r^2\dot{\theta}^2 + \dot{\theta}^2\right)\right)}$$

$$[a, \omega] = 0$$

Out[54]=

$$[a, T] = -\Bigg(\Bigg(2\, r^4\, td^2\, (rd^2 + r^2\, td^2)^2 \Bigg(-k^2\, rd\sqrt{\frac{k^2 + m\, r^4\, rd^2\, td^2 + m\, r^6\, td^4}{k^2}} + rd\,(k^2 + m\, r^4\, td^2\,(rd^2 + r^2\, td^2))\,\mathrm{Cos}\Bigg[2\,\mathrm{ArcTan}\Bigg[\frac{\left(1 - \sqrt{1 + \frac{m\, r^4\, td^2\,(rd^2+r^2\, td^2)}{k^2}}\right)\mathrm{Tan}[\frac{t}{2}]}{1 + \sqrt{1 + \frac{m\, r^4\, td^2\,(rd^2+r^2\, td^2)}{k^2}}}\Bigg]\Bigg] + r\, td\Bigg(-2\, k^2\, \mathrm{Sin}[t] + 2\, k^2\sqrt{\frac{k^2 + m\, r^4\, rd^2\, td^2 + m\, r^6\, td^4}{k^2}}\,\mathrm{Sin}\Bigg[t - 2\,\mathrm{ArcTan}\Bigg[\frac{\left(1 - \sqrt{1 + \frac{m\, r^4\, td^2\,(rd^2+r^2\, td^2)}{k^2}}\right)\mathrm{Tan}[\frac{t}{2}]}{1 + \sqrt{1 + \frac{m\, r^4\, td^2\,(rd^2+r^2\, td^2)}{k^2}}}\Bigg]\Bigg]\Bigg) - (2\, k^2 + m\, r^4\, rd^2\, td^2 + m\, r^6\, td^4)\,\mathrm{Sin}\Bigg[2\,\mathrm{ArcTan}\Bigg[\frac{\left(1 - \sqrt{1 + \frac{m\, r^4\, td^2\,(rd^2+r^2\, td^2)}{k^2}}\right)\mathrm{Tan}[\frac{t}{2}]}{1 + \sqrt{1 + \frac{m\, r^4\, td^2\,(rd^2+r^2\, td^2)}{k^2}}}\Bigg]\Bigg]\Bigg)\Bigg)\Bigg/\Bigg(k^3\left(-\frac{(rd^2 + r^2\, td^2)^3}{k^2\, m}\right)^{3/2}\sqrt{\frac{k^2 + m\, r^4\, rd^2\, td^2 + m\, r^6\, td^4}{k^2}}\left(2\, k^2 + m\, r^4\, td^2\,(rd^2 + r^2\, td^2) + 2\, k^2\sqrt{\frac{k^2 + m\, r^4\, rd^2\, td^2 + m\, r^6\, td^4}{k^2}}\,\mathrm{Cos}[t]\right)\Bigg)\Bigg)$$

where rd represent $\dot{r}$, td represent $\dot{\theta}$, t represent θ.

$$[e, \psi] = \frac{m^2r^8\dot{r}\dot{\theta}^4\left(r^2\dot{\theta}^2 + \dot{r}^2\right)}{k^2\sqrt{\frac{k^2+mr^6\dot{\theta}^4+mr^4\dot{r}^2\dot{\theta}^2}{k^2}}\left(2k^2\cos\theta\sqrt{\frac{k^2+mr^6\dot{\theta}^4+mr^4\dot{r}^2\dot{\theta}^2}{k^2}} + 2k^2 + mr^4\dot{\theta}^2\left(r^2\dot{\theta}^2 + \dot{r}^2\right)\right)}$$

$$[e,\omega] = \frac{3mr^5\dot{\theta}^3\sqrt{-\frac{\left(r^2\dot{\theta}^2+\dot{r}^2\right)^3}{k^2m}}}{k^2\sqrt{\frac{k^2+mr^6\dot{\theta}^4+mr^4\dot{r}^2\dot{\theta}^2}{k^2}}}$$

$$[\psi,\omega] = -\frac{3mr^4\dot{\theta}^2\sqrt{-\frac{\left(r^2\dot{\theta}^2+\dot{r}^2\right)^3}{k^2m}}\left(\dot{r}\sqrt{\frac{k^2+mr^6\dot{\theta}^4+mr^4\dot{r}^2\dot{\theta}^2}{k^2}}+2r\dot{\theta}\sin\theta\right)}{\sqrt{\frac{k^2+mr^6\dot{\theta}^4+mr^4\dot{r}^2\dot{\theta}^2}{k^2}}\left(2k^2\cos\theta\sqrt{\frac{k^2+mr^6\dot{\theta}^4+mr^4\dot{r}^2\dot{\theta}^2}{k^2}}+2k^2+mr^4\dot{\theta}^2\left(r^2\dot{\theta}^2+\dot{r}^2\right)\right)}$$

$$[\omega,T] = \frac{-\left(3 m r^4 \mathrm{td}^2\left(-k^2 \mathrm{rd}\sqrt{\frac{k^2+m r^4 \mathrm{rd}^2 \mathrm{td}^2+m r^6 \mathrm{td}^4}{k^2}}+\mathrm{rd}\left(k^2+m r^4 \mathrm{td}^2\left(\mathrm{rd}^2+r^2 \mathrm{td}^2\right)\right)\mathrm{Cos}\left[2\,\mathrm{ArcTan}\left[\frac{\left(1-\sqrt{1+\frac{m r^4 \mathrm{td}^2\left(\mathrm{rd}^2+r^2 \mathrm{td}^2\right)}{k^2}}\right)\mathrm{Tan}\left[\frac{t}{2}\right]}{1+\sqrt{1+\frac{m r^4 \mathrm{td}^2\left(\mathrm{rd}^2+r^2 \mathrm{td}^2\right)}{k^2}}}\right]\right]+r\,\mathrm{td}\left(-2k^2\mathrm{Sin}[t]+2k^2\sqrt{\frac{k^2+m r^4 \mathrm{rd}^2 \mathrm{td}^2+m r^6 \mathrm{td}^4}{k^2}}\,\mathrm{Sin}\left[t-2\,\mathrm{ArcTan}\left[\frac{\left(1-\sqrt{1+\frac{m r^4 \mathrm{td}^2\left(\mathrm{rd}^2+r^2 \mathrm{td}^2\right)}{k^2}}\right)\mathrm{Tan}\left[\frac{t}{2}\right]}{1+\sqrt{1+\frac{m r^4 \mathrm{td}^2\left(\mathrm{rd}^2+r^2 \mathrm{td}^2\right)}{k^2}}}\right]\right]-\left(2k^2+m r^4 \mathrm{rd}^2 \mathrm{td}^2+m r^6 \mathrm{td}^4\right)\mathrm{Sin}\left[2\,\mathrm{ArcTan}\left[\frac{\left(1-\sqrt{1+\frac{m r^4 \mathrm{td}^2\left(\mathrm{rd}^2+r^2 \mathrm{td}^2\right)}{k^2}}\right)\mathrm{Tan}\left[\frac{t}{2}\right]}{1+\sqrt{1+\frac{m r^4 \mathrm{td}^2\left(\mathrm{rd}^2+r^2 \mathrm{td}^2\right)}{k^2}}}\right]\right]\right)\right)\right)}{\left(k^2\sqrt{\frac{k^2+m r^4 \mathrm{rd}^2 \mathrm{td}^2+m r^6 \mathrm{td}^4}{k^2}}\left(2k^2+m r^4 \mathrm{td}^2\left(\mathrm{rd}^2+r^2 \mathrm{td}^2\right)+2k^2\sqrt{\frac{k^2+m r^4 \mathrm{rd}^2 \mathrm{td}^2+m r^6 \mathrm{td}^4}{k^2}}\,\mathrm{Cos}[t]\right)\right)}$$

where rd represent $\dot{r}$, td represent $\dot{\theta}$, t represent θ.

$$[\psi,T] = \text{very very complicated}\dots$$

$$[e,T] = \text{very very complicated}\dots$$

9.30:

(a) Let's say A and B are 2 constants of motion, we can prove that,

$$[[A,B],H] = [AB,H] - [BA,H] = A[B,H] + [A,H]B - B[A,H] - [B,H]A = 0.$$

Therefore, $[A,B]$ is a constant of motion even when the constants depend upon time explicitly.
(b) We can show that, if,

$$[F,H] = \frac{\partial F}{\partial t},$$

$$[H,H] = \frac{\partial H}{\partial t} = 0,$$

then,

$$[\frac{\partial^n F}{\partial t^n}, H] = \frac{\partial^n}{\partial t^n}[F,H] = \frac{\partial}{\partial t}\frac{\partial^n F}{\partial t^n}.$$

Therefore, the nth derivative of F is a constant of motion. **(c)** In this case, the Hamiltonian,

$$H = \frac{p^2}{2m}.$$

The Poisson bracket,

$$[H, F] = \frac{\partial H}{\partial x}\frac{\partial F}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial F}{\partial x} = -\frac{p}{m},$$

and the partial derivative of F is,

$$\frac{\partial F}{\partial t} = -\frac{p}{m} = [H, F].$$

Therefore, F is a constant of motion.