Independence of general coordinate and velocity in Lagrangian

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1 Newtonian point of view

In the Newtonian mechanics, it is declared that,

$$\vec{F} = m\ddot{\vec{x}}.$$

In this formula, F is determined by the nature of the force, which is not a new quantity but a specific function of coordinate $\vec{x}$ and velocity $\dot{\vec{x}}$ case by case (which is a reasonable assumption that the force is not dependent on higher derivatives of position). Therefore

$$F(\vec{x}, \dot{\vec{x}}) = m\ddot{\vec{x}}$$

is purely a second-order ordinary differential equation. To solve this equation, we need two initial condition which are initial position $\vec{x}(0)$ and velocity $\dot{\vec{x}}(0)$ to determine the motion of a particle. Therefore, we can conclude that position and velocity are actually two independent parameters when determining a motion, which means each combination $(\vec{x}(0), \dot{\vec{x}}(0))$ may lead a totally different motion. But as long as the motion is determined, the position and velocity is no longer independent. As we know the specific form of position $\vec{x}(t)$, the velocity is just its derivative with respect to time $\vec{v}(t) = \dot{\vec{x}}(t)$.

2 Lagrangian point of view

We can use the same idea in Lagrangian mechanics. In Lagrangian mechanics to determine a state, we need points like $(q_i, \dot{q}_i)$ in a configuration space (again we don't consider higher derivatives of position), which q_i is a general coordinate and $\dot{q}_i$ is a general velocity, the index i runing from 1 to s, s is the degree of freedom of the system. Then every point in the configuration space represent a state, and the evolution of state by time constructed a line in the configuration space. We call the process of evolution by time as a motion. Obviously from the initial state to final state

$$(q_i, \dot{q}_i) \quad \rightarrow \quad (q_f, \dot{q}_f).$$

There are infinite lines that could connect between two states, thus infinite potential "motions" between to state. However, we can define a quantity $\mathscr{L}(q_i, \dot{q}_i, t)$ called Lagrangian, which can uniquely determine one path as the real motion. The real path should satisfy that $\int_i^f \mathscr{L}(q_i, \dot{q}_i, t)dt$ takes extrema. Equivalently,

$$\delta \int_i^f \mathscr{L}(q_i, \dot{q}_i, t)dt = 0.$$

We can also conclude that before the real motion is determined, there are infinite paths that connected the states, therefore the universal Lagrangian $\mathscr{L}$ should be a function of q_i, $\dot{q}_i$ and t. Of course, q_i and $\dot{q}_i$ is independent for the nature of the configuration space (it is a 2s-dimension space). Also, as long as the motion is determined, we have pick a unique path in the path bundle. Then the path is a unique line in the configuration space, any position correspond a value of velocity, hence they are not independent anymore.

More mathematically, the configuration space is actually a differentiable manifold M, consideing its tangent bundle TM which contains all possible tangent vector on the manifold, each vector represent a state $(q_i, \dot{q}_i)$. Then Lagrangian is a map $L : TM \times \mathbb{R} \to \mathbb{R}$, which also indicate that general coordinate and velocity are independent. Only when the specific condition the Lagrangian satisfied, can we determine the motion.