A problem about rigid body

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Problem

四、（40 分）如图 4 所示，半径为R、质量均为m的均匀实心球体 A、B、C 两两相切地放置在光滑水平面上。三个球体的外侧有一光滑固定圆筒，其内表面与 A、B、C 外切。初始时刻，A、B、C 静止，一绕圆筒竖直中心轴转动的均匀球体 D 轻放于球体 A、B、C 正上方。已知 D 的质量也为m、半径也为R，D 的初始角速度大小为Ω_0，D 与 A、B、C 之间的动摩擦因数为μ，A、B、C 三个球体之间无摩擦。经历一段时间后，整个体系达到 D 相对于 A、B、C 没有滑动的稳定状态。重力加速度大小为g。求：

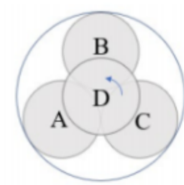

图 4

（1）体系从初始状态到达稳定状态所需时间，以及稳定状态时，球体 A、D 的角速度大小。

（2）达到稳定状态时，A、C 两球切点的相对速度大小与 A 球质心速度大小之比。

Solution:

Firstly, we write down the Lagrangian of each ball, for the top ball

$$L = \frac{1}{2} I\omega^2 - mg\frac{2\sqrt{3}}{3}R,$$

and for the other 3 bottom balls which are symmetric,

$$L' = \frac{1}{2}mr^2\Omega^2 + \frac{1}{2}I(\omega_1^2 + \omega_2^2 + \omega_3^2),$$

where $I = \frac{2}{5}mR^2$ and $r = \frac{2\sqrt{3}}{3}R$. Then by the Euler's equations, we have

$$\dot{L} = M + L \times \Omega,$$

that are

$$I\dot{\omega_1} = f_1 R\cos\theta + I\omega_2\Omega, \quad \text{(x direction)} \tag{1}$$

$$I\dot{\omega_2} = f_2 R - I\omega_1\Omega, \quad \text{(y direction)} \tag{2}$$

$$I\dot{\omega_3} = f_1 R\sin\theta, \quad \text{(z direction)} \tag{3}$$

where θ is half of the top angle of the tetrahedron satisfing $\sin\theta = \frac{\sqrt{3}}{3}$ and $\cos\theta = \frac{\sqrt{6}}{3}$, f_1 and f_2 are respectively the horizontal and vertical component of friction.

Secondly, using the Lagrangian's equation, we can derive

$$mg = 3N\cos\theta + 3f_2\sin\theta, \quad \text{(z direction equilibrium for the top ball)} \tag{4}$$

$$\dot{\omega} = -3f_1 R\sin\theta, \quad \text{(rotation equation for the top ball)} \tag{5}$$

$$mr\dot{\Omega} = f_1. \quad \text{(rotation equation for the system)} \tag{6}$$

We can calculate the relative velocity of the meeting point. Denoting its horizontal and vertical component as v_1 and v_2 respectively, we have

$$v_1 = (\Omega r - \omega_1 R\cos\theta - \omega_3 R\sin\theta) - \omega R\sin\theta,$$

and

$$v_2 = \omega_2 R.$$

Then we have the constraint on the friction

$$\frac{f_1}{f_2} = \frac{v_1}{v_2} = \frac{(\Omega r - \omega_1 R\cos\theta - \omega_3 R\sin\theta) - \omega R\sin\theta}{\omega_2 R}, \tag{7}$$

and the relationship between friction and constraining force

$$f = \sqrt{f_1^2 + f_2^2} = \mu N. \tag{8}$$

Now we have 8 unknowns, $f_1, f_2, N, \omega, \omega_1, \omega_2, \omega_3, \Omega$ and 8 equations. It is worth noting that the above discussion is under the premise of balls slipping on the meeting point.

We then solve these equations numerically. We set the initial condition as

$$m = 5kg,$$

$$g = 9.8m/s^2,$$

$$R = 0.2m,$$

$$\mu = 0.1,$$

$$\omega_0 = 7s^{-1},$$

$$\omega_{10} = \omega_{20} = \omega_{30} = \Omega_0 = 0.$$

We first consider the evolution of angular velocity with slipping. After that we can consider the pure rolling condition and set a cut-off on the results.

we can use this code in Fig. 1 of Mathematica to numerically solve angular velocities.

```
In[1]:= ClearAll["`*"]
m = 5;
g = 9.8;
R = 0.2;
mu = 0.1;
w = 7;
T = 20;
tabt = Table[0.05 x, {x, 0, 20 / 0.05}];
eqns = {
  2/5 m R w1'[t] == f1[t] Sqrt[6]/3 + 2/5 m R w2[t] Omega[t], (*小球转动方程*)
  2/5 m R w2'[t] == f2[t] - 2/5 m R w1[t] Omega[t], (*小球转动方程*)
  2/5 m R w3'[t] == f1[t] Sqrt[3]/3, (*小球转动方程*)
  m g == N0[t] Sqrt[6] + f2[t] Sqrt[3], (*竖直方向平衡*)
  w0'[t] == -Sqrt[3] f1[t] R, (*大球转动方程*)
  2/3 Sqrt[3] m R Omega'[t] == f1[t], (*系统进动方程*)
  f1[t] w2[t] == (Omega[t] 2/3 Sqrt[3] - w1[t] Sqrt[6]/3 - w3[t] Sqrt[3]/3 - w0[t] Sqrt[3]/3) f2[t], (*约束方程*)
  Sqrt[f1[t]^2 + f2[t]^2] == mu N0[t], (*摩擦力与支持力关系*)
  w0[0] == w, w1[0] == 0, w2[0] == 0, w3[0] == 0, Omega[0] == 0};

sol = NDSolve[{eqns}, {f1[t], f2[t], N0[t], w0[t], w1[t], w2[t], w3[t], Omega[t]}, {t, 0, T}];

Plot[w0[t] /. sol, {t, 0, T}]
tab0 = Table[w0[t] /. sol, {t, 0, T, 0.05}];
map0 = Transpose@{tabt, tab0} // TableForm
Plot[w1[t] /. sol, {t, 0, T}]
tab1 = Table[w1[t] /. sol, {t, 0, T, 0.05}];
Plot[w2[t] /. sol, {t, 0, T}]
tab2 = Table[w2[t] /. sol, {t, 0, T, 0.05}];
Plot[w3[t] /. sol, {t, 0, T}]
tab3 = Table[w3[t] /. sol, {t, 0, T, 0.05}];
Plot[Omega[t] /. sol, {t, 0, T}]
tabO = Table[Omega[t] /. sol, {t, 0, T, 0.05}];
```

Figure 1: The code of calculating slipping state

The numerical result is showed in the following Fig. 2,

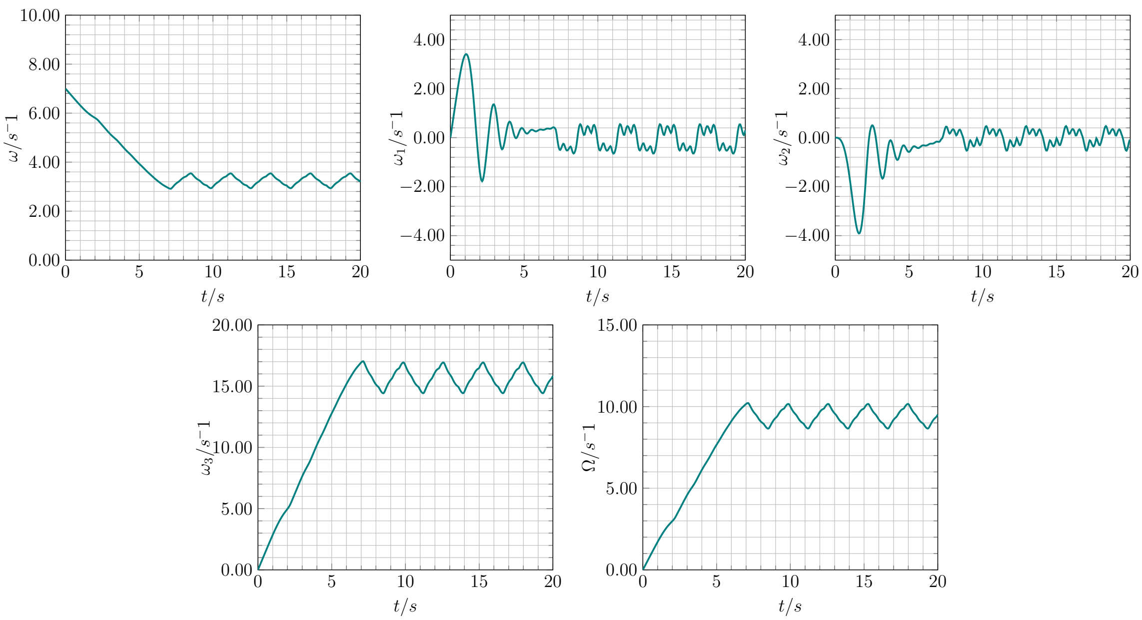

Figure 2: The numerical results of angular velocities with slipping

Then we can combine them together as one and do some mathematical treatment as shown in Fig. 3. By this figure, we can see the condition for pure rolling.

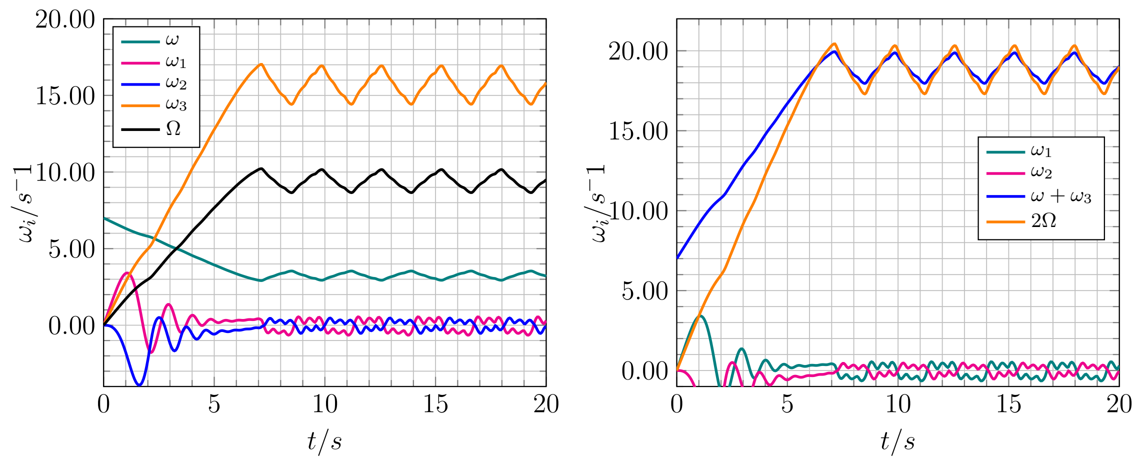

Figure 3: The numerical results of angular velocities combining in one figure

The pure rolling condition are $v_1 = 0$ and $v_2 = 0$, that mean $\omega_1 = \omega_2$ and $2\Omega = \omega + \omega_3$. From the above figure, we can point out that the condition roughly satisfied occasionally, such as $t = 7.2 \sim 7.6s$ or $t = 10 \sim 10.4s$. Providing the time long enough, we can finally get some point that the system stop slipping. Let's say it happens at $t = 7.25s$, then we cut off the value after that moment and get the final result as following Fig. 4. In fact, ω_1 and ω_2 should both be 0 after cutting-off, but we allow a little error here. As we mentioned before, the pure rolling condition is roughly satisfied, not precisely.

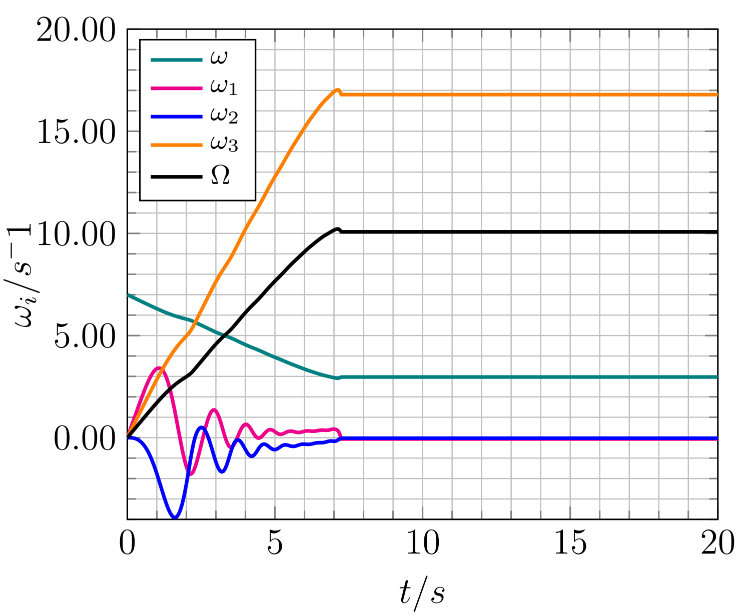

Figure 4: The numerical results of angular velocities after cutting-off