Group theory & Dirac equation.

Klein - Gordon $\quad (\Box + m^2)\psi = 0 \quad$ spin - 0

$E^2 - p^2 = m^2 \longrightarrow E = \sqrt{p^2 + m^2}. \quad$ spin - $\frac{1}{2}$

$E \to i\frac{\partial}{\partial t} \quad p = -i\nabla$

$$i\frac{\partial}{\partial t}\psi = \sqrt{-\nabla^2 + m^2}\ \psi$$

Def : $(G, \cdot)$

- closure : $g_1, g_2 \in G$, $g_1 \cdot g_2 \in G$.
- Identity , $\exists e \in G$, $g \cdot e = e \cdot g = g$.
- Inverse , $\forall g \in G$, $\exists g^{-1} \in G$, $g \cdot g^{-1} = g^{-1} g = e$.
- Associativity : $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$

Discrete / continuous (Lie)

Isomorphism

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \overbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}^{R} \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow SO(2)$$

$\det R = 1$

$\begin{pmatrix} +1 & \\ & -1 \end{pmatrix}$

$$x' + iy' = (x + iy)\, e^{i\theta} \longrightarrow U(1)$$

$U U^{+} = 1$

$SO(2) \cong U(1)$

$e^{i\theta} e^{-i\theta} = 1$

SO(3) R

$r'^2 = r^2 \Rightarrow \quad r' r'^T = r r^T = r^T R^T R r = r^T r$

$r' = R r \qquad \rightarrow \quad \underline{R^T R = 1}$ orthogonal.

$$(R_1 R_2)^T (R_1 R_2) = R_2^T R_1^T R_1 R_2 = 1$$

All orthogonal 3×3 matrix, $\det R = 1$, SO(3)

not - Abelian

Degree of freedom $\qquad R^T R$

$$\underline{R^T R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$9 - 6 = 3$

$$R_z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

$R_y(\theta)$

$$= \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$J_z = \frac{1}{i} \left.\frac{dR_z}{d\theta}\right|_{\theta=0} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$J_x = \frac{1}{i}\left.\frac{dR_x}{d\theta}\right|_{\theta=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$J_y = \frac{1}{i}\left.\frac{dR_y}{d\theta}\right|_{\theta=0} = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$$

$\Rightarrow [J_i, J_j] = i\epsilon_{ijk} J_k$

$$\underline{[J_x, J_y] = iJ_z.}$$

$\downarrow$

Lie algebra.

$$R_z(\delta\theta) = 1 + \left.\frac{dR_z}{d\theta}\right|_{\theta=0}\delta\theta$$

$$= 1 + iJ_z\delta\theta$$

$$R_z(\theta) = \lim_{N\to\infty}\left[R_z\left(\frac{\theta}{N}\right)\right]^N = \left[1 + iJ_z\frac{\theta}{N}\right]^N = e^{iJ_z\theta}$$

$R_x(\theta) = e^{iJ_x\theta}$, $R_y(\theta) = e^{iJ_y\theta}$.

$$R_{\vec{n}}(\theta) = e^{i\vec{J}\cdot\vec{\theta}} = e^{i\vec{J}\cdot\vec{n}\,\theta}$$

SU(2) $\quad U^\dagger U = 1 \longrightarrow \underline{U^\dagger = U^{-1}} \quad \det U = 1$

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \underline{\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}}. \quad |a|^2 + |b|^2 = 1$$

$$\Rightarrow \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \Rightarrow a^* = d,\ b^* = -c$$

Pauli spinor $\quad \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \quad \xi \to U\xi, \quad \xi^\dagger = \xi^\dagger U^\dagger$

$(\xi_1^*\ \xi_2^*)$

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \qquad \begin{pmatrix} -\xi_2^* \\ \xi_1^* \end{pmatrix}$$

$$\begin{cases} \xi_1' = a\xi_1 + b\xi_2 \\ \xi_2' = -b^*\xi_1 + a^*\xi_2 \end{cases} \Rightarrow \begin{cases} \xi_1^{*\prime} = a^*\xi_1^* + b^*\xi_2^* \\ \xi_2^{*\prime} = -b\xi_1^* + a^*\xi_2^* \end{cases}$$

$$\begin{cases} -\xi_2'^* = \underline{a}(-\xi_2^*) + \underline{b}\,\xi_1^* \\ \xi_1'^* = -\underline{b}^*(-\xi_2) + \underline{a}^*\,\xi_1^* \end{cases}$$

$$\begin{pmatrix} -\xi_2^* \\ \xi_1^* \end{pmatrix} = \underline{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} \begin{pmatrix} \xi_1^* \\ \xi_2^* \end{pmatrix} = \varsigma\,\xi_1^*$$

$$\xi_1 \sim \varsigma\,\xi_1^*$$

$$\underline{\xi_1^\dagger} \sim (\varsigma\,\xi)^T = \underline{(-\xi_2, \xi_1)}$$

$$\mathcal{H} = \xi\xi^\dagger = \cancel{(-\xi_1}$$

$$\sim \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} (-\xi_2, \xi_1) = \underline{\begin{pmatrix} -\xi_1\xi_2 & \xi_1^2 \\ -\xi_2^2 & \xi_1\xi_2 \end{pmatrix}} \text{ traceless.}$$

$$h = \vec{\sigma}\cdot\vec{r} = \underline{\begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix}}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$x = \frac{1}{2}(\xi_2^2 - \xi_1^2), \quad y = \frac{1}{2i}(\xi_1^2 + \xi_2^2), \quad z = \xi_1\xi_2.$$

$$h \to UhU^\dagger = h' \qquad \det h = x^2 + y^2 + z^2$$

$$UhU^{-1}$$

Degree of freedom $4 - 1 = 3$

$b = 0$, $a = e^{i\alpha/2}$

$$U = \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix} \longrightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xi_1' = e^{i\alpha/2}\,\xi_1 + 0$$

$$\xi_2' = e^{-i\alpha/2}\,\xi_2 + 0.$$

$$U = e^{i\sigma_z \alpha/2} \longleftrightarrow e^{iJ_z\alpha}. \qquad \vec{\sigma}\cdot\vec{n} = I$$

$$U = e^{i\vec{\sigma}\cdot\vec{\theta}/2} = e^{i\vec{\sigma}\cdot\vec{n}\,\theta/2} \qquad \cancel{(\vec{\sigma}\cdot\vec{\theta})^2 = (\vec{\theta}\cdot\vec{\theta})\,I} = \cancel{\theta}.$$

$$= \cancel{1 + i(\vec{\sigma}\cdot\vec{\theta})/2 - \frac{(\vec{\sigma}\cdot\vec{\theta})^2}{8}} + \cdots$$

$$= 1 + i\vec{\sigma}\cdot\vec{n}\,\frac{\theta}{2} - \frac{(\vec{\sigma}\cdot\vec{n})^2}{8}\,\theta^2 + \cdots$$

$$= i(\vec{\sigma}\cdot\vec{n})\sin\frac{\theta}{2} + \cos\frac{\theta}{2} \longleftrightarrow e^{i\vec{J}\cdot\vec{\theta}}$$

$\alpha = 2\pi$, $U \to -U$, $R \to R$.

$$U \to R \qquad SO(3) \cong SU(2)/Z_2.$$

$$-U \to$$

$\uparrow$ double cover of $SO(3)$

$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = i\,\epsilon_{ijk}\,\frac{\sigma_k}{2}$$

$$so(3) = su(2)$$

Casimir element

$$J^2 \to j(j+1)$$

j^2

SL(2, C) & Lorentz group

$$\begin{pmatrix} x^{0\prime} \\ x^{1\prime} \\ x^{2\prime} \\ x^{3\prime} \end{pmatrix} = \begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\cosh\phi = \gamma$

$\sinh\phi = \gamma\beta.$

$$K_x = \frac{1}{i}\frac{\partial B_x}{\partial\phi}\Big|_{\phi=0} = -i\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

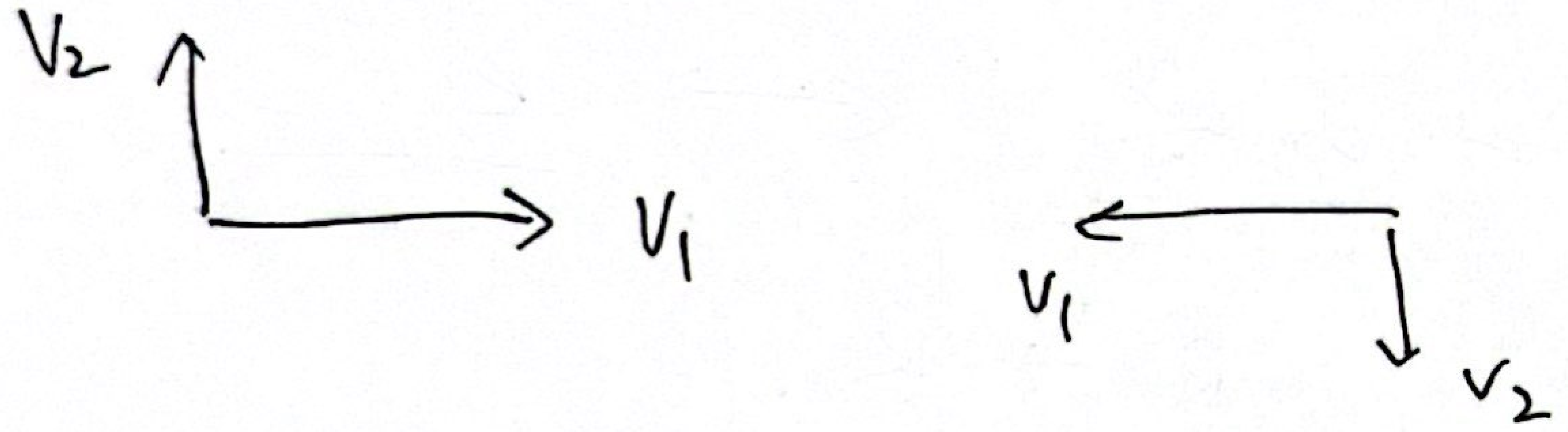
$$K_y = -i\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_z = -i\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\left\{\begin{aligned} [J_i, J_j] &= i\epsilon_{ijk} J_k \\ [J_i, K_j] &= i\epsilon_{ijk} K_k \\ [K_i, K_j] &= -i\epsilon_{ijk} J_k \end{aligned}\right.$$

$\vec{J} \sim \frac{\vec{\sigma}}{2}$

$K \sim i\frac{\vec{\sigma}}{2}$



$$e^{iK_x\delta\phi}\, e^{iK_y\delta\phi'}\, e^{-iK_x\delta\phi}\, e^{-iK_y\delta\phi'}$$

$[K_x, K_y] = -iJ_z.$

$$= 1 - [K_x, K_y]\,\delta\phi\,\delta\phi'$$

$$= 1 + iJ_z\,\delta\phi\,\delta\phi'$$

$\vec{A} = \frac{1}{2}(\vec{J} + i\vec{K})$ $\quad$ $\vec{B} = \frac{1}{2}(\vec{J} - i\vec{K})$

$$\left\{\begin{aligned} [A_x, A_y] &= \frac{1}{4}[J_x + iK_x, J_y + iK_y] = iA_z. \\ [B_x, B_y] &= iB_z \end{aligned}\right.$$

$[A_i, B_j] = 0.$

$$so(1,3;\mathbb{C}) = su(2) \oplus su(2)$$

j $\quad j'$

A $\quad$ B

(j, j')

$(j,0)$: $\vec{J} = i\vec{K}$

$(0,j)$: $\vec{J} = -i\vec{K}$

$(\frac{1}{2},0)$: $\vec{J} = \frac{\vec{\sigma}}{2}$ $\quad \vec{K} = -i\vec{\sigma}/2$ $\quad \exp(i\frac{\sigma}{2}\theta + \frac{\sigma}{2}\phi) = M$

$(0,\frac{1}{2})$: $\vec{J} = \vec{\sigma}/2$ $\quad \vec{K} = i\vec{\sigma}/2$ $\quad \exp(i\frac{\sigma}{2}(\theta + i\phi)) = N$

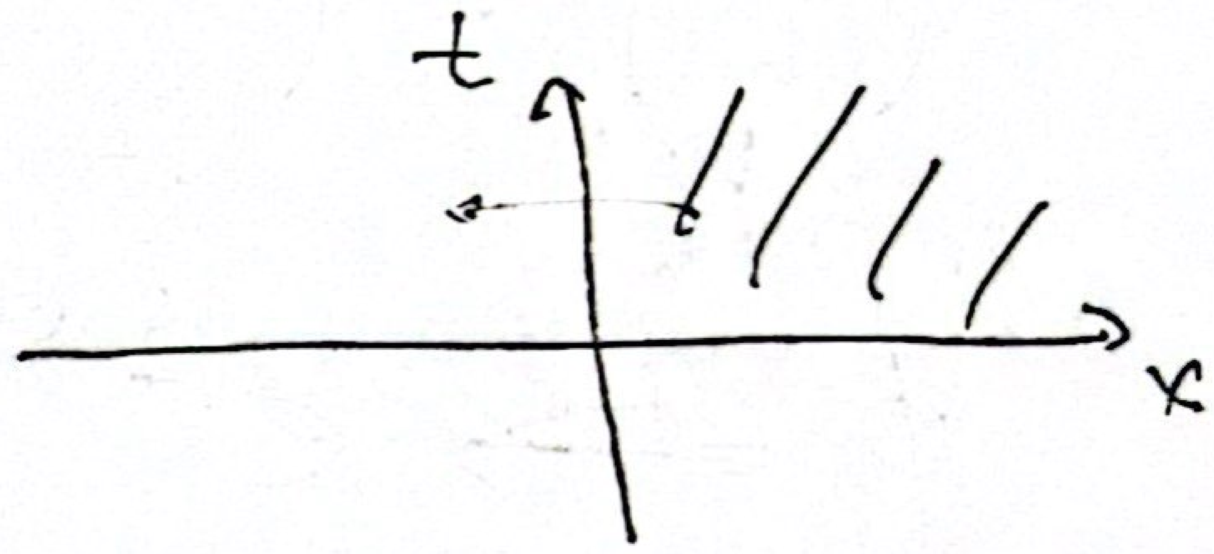
$$N = \varsigma M^* \varsigma^{-1}, \quad \varsigma = -i\sigma_2.$$

$$\underline{SL(2;\mathbb{C})} = \{ M \in GL(2,\mathbb{C}),\ \det(M) = 1 \}.$$

Degree of freedom

$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad ad - bc = 1$

$6 \left\{ \begin{array}{ll} 3 & \text{Euler} \\ 3 & \text{velocity} \end{array} \right.$



Parity operation $\quad P \quad \vec{v} \to -\vec{v}$

$\vec{K} \to -\vec{K}$, $\quad \vec{J} \to \vec{J}$

vector $\qquad$ pseudo vector $\qquad \psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$

Dirac spinor

$(j,0) \to (0,j)$

$\xi \to \eta$

$$\begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = \begin{pmatrix} e^{\frac{i}{2}\sigma(\theta - i\phi)} & 0 \\ 0 & e^{\frac{i}{2}\sigma(\theta + i\phi)} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

$$= \begin{pmatrix} D(\Lambda) & 0 \\ 0 & \bar{D}(\Lambda) \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} X^{\nu} \qquad P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$\underline{e^{\frac{1}{2}\sigma\phi}}$ Lorentz group is not compact.

$\theta \in [0, 2\pi]$

$\frac{v}{c} \in [0, 1)$

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \phi_R \\ \phi_L \end{pmatrix} \qquad \cosh(\phi/2) = \left[\frac{\gamma+1}{2}\right]^{1/2}$$

$$\sinh(\phi/2) = \left[\frac{\gamma-1}{2}\right]^{1/2}$$

$$\phi_R \Longrightarrow e^{1/2\,\sigma\phi}\,\phi_R.$$

$$= [\cosh(\phi/2) + \vec{\sigma}\cdot\vec{n}\sinh(\phi/2)]\,\phi_R.$$

$$\phi_R(\vec{p}) = \left[\left[\frac{\gamma+1}{2}\right]^{1/2} + \vec{\sigma}\cdot\hat{p}\left(\frac{\gamma-1}{2}\right)^{1/2}\right]\phi_R(0) \qquad \underline{\gamma = E/m.}$$

$$E = \gamma m c^2$$

$$= \frac{E + m + \vec{\sigma}\cdot\vec{p}}{[2m(E+m)]^{1/2}}\,\phi_R(0)$$

$$\phi_L(\vec{p}) = \frac{E + m - \vec{\sigma}\cdot\vec{p}}{[2m(E+m)]^{1/2}}\,\phi_L(0).$$

$$\phi_R(0) = \pm\phi_L(0)$$

$$\phi_R(\vec{p})(E+m-\vec{\sigma}\cdot\vec{p}) = \phi_L(\vec{p})(E+m+\vec{\sigma}\cdot\vec{p}).$$

$$\Rightarrow \quad \phi_R(\vec{p}) = \frac{E+\vec{\sigma}\cdot\vec{p}}{m}\phi_L(\vec{p})$$

$$\phi_L(\vec{p}) = \frac{E-\vec{\sigma}\cdot\vec{p}}{m}\phi_R(\vec{p}).$$

$$\begin{cases} -m\phi_R(\vec{p}) + (p_0+\vec{\sigma}\cdot\vec{p})\phi_L(\vec{p}) = 0 \\ (p_0-\vec{\sigma}\cdot\vec{p})\phi_R(\vec{p}) - m\phi_L(\vec{p}) = 0. \end{cases}$$

$$\begin{pmatrix} -m & p_0+\vec{\sigma}\cdot\vec{p} \\ p_0-\vec{\sigma}\cdot\vec{p} & -m \end{pmatrix} \underline{\begin{pmatrix} \phi_R(\vec{p}) \\ \phi_L(p) \end{pmatrix}} = 0$$

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

$$(\gamma^0 p_0 + \gamma^i p_i - m)\psi(p) = 0.$$

$$= (\gamma^\mu p_\mu - m)\psi(p) = 0, \qquad \gamma^\mu a_\mu = \not{a}.$$

$$(\not{p} - m)\psi(p) = 0$$

$$\begin{cases} (p_0+\vec{\sigma}\cdot\vec{p})\phi_L(\vec{p}) = 0 \\ (p_0-\vec{\sigma}\cdot\vec{p})\phi_R(\vec{p}) = 0 \end{cases} \qquad \text{Weyl equation.}$$

$$\vec{\sigma}\cdot\vec{p}\,\phi_L = -|\vec{p}|\phi_L \Rightarrow \vec{\sigma}\cdot\hat{p}\,\phi_L = -\phi_L$$

$$\vec{\sigma}\cdot\hat{p}\,\phi_R = \phi_R$$