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Poincaré group

① $[S_i, S_j] = i\epsilon_{ijk} S_k$ $\quad \vec{S}\cdot\vec{S} = S^2 = s(s+1)$

② $[\vec{S}, \not{p}] = 0$ $\quad (\not{p} - m)u = 0$ $\quad \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix}$

$\not{p} u = m u$

$\vec{S} = \frac{1}{2}\vec{\Sigma} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$ $\quad [\vec{S}, \not{p}] \neq 0$

$\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu$

$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$

$\gamma_i \gamma_j$

$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$

$\underline{\sigma^{ij}} = \frac{i}{2}[\gamma^i, \gamma^j]$

$= -\frac{i}{2}\begin{pmatrix} [\sigma_i, \sigma_j] & 0 \\ 0 & [\sigma_i, \sigma_j] \end{pmatrix}$

$= \epsilon_{ijk}\begin{pmatrix} \sigma_k & \\ & \sigma_k \end{pmatrix} = \epsilon_{ijk}\underline{\Sigma_k}$ $\quad [\vec{\Sigma}, \gamma^\mu] \neq 0$

$\Sigma_k = \frac{1}{2}\epsilon_{ijk}\sigma^{ij}$ $\quad [\Sigma_k, \gamma^\mu] = \frac{1}{2}\epsilon_{ijk}\frac{i}{2}\Big[[\gamma^i, \gamma^j], \gamma^\mu\Big]$

$[AB, C] = A\{B, C\} - \{A, C\}B$

$= \frac{i}{4}\epsilon_{ijk}\Big\{[\gamma^i\gamma^j, \gamma^\mu] - [\gamma^j\gamma^i, \gamma^\mu]$

$= \frac{i}{4}\epsilon_{ijk}\Big\{\underline{\gamma^i \cdot 2g^{j\mu}} - \underline{2g^{i\mu}\gamma^j} - \underline{2\gamma^j g^{i\mu}} + \underline{\gamma^i \cdot 2g^{j\mu}}\Big\}$

$= i\epsilon_{ijk}(\gamma^j g^{i\mu} - \gamma^i g^{j\mu}) \neq 0$

unless $\quad [\Sigma_k, \gamma^0] = 0$

$\not{p} = \gamma^0 E$

$[J^2, K_1] \neq 0$

$[J_1^2 + J_2^2 + J_3^2, K_1]$ $\qquad J \pm iK$

Spin

mass $\longrightarrow M^2 = P^\mu P_\mu$ $\qquad P^\mu : x^\mu \rightarrow x'^\mu = x^\mu + a^\mu$

$M^2 > 0$ $\qquad$ Inhomogeneous Lorentz group

$M^2 = 0$, helicity $\qquad$ Poincaré group.

$M^2 < 0$ $\qquad ISO(1,3)$

Wigner

$$e^{iP \cdot a} \rightarrow e^{iK\phi} \rightarrow e^{-iPa} \rightarrow e^{-iK\phi}$$

$$(1 - iK\phi)(1 - iPa)(1 + iK\phi)(1 + iPa)$$

$$= 1 + [P_\mu, P_\nu] a^\mu a^\nu + 2[P^\mu, K_i] a^\mu \phi_i + [K_i, K_j]\phi_i \phi_j$$

Lie algebra.

3 J $\qquad J_k = \epsilon_{ijk} x_i P_j = -i \epsilon_{ijk} x_i \partial_j$

$[J_x, J_y] = i J_z$

3 K $\qquad$ generator

4 P $\qquad X_\alpha = i \left.\frac{\partial x'^\mu}{\partial a^\alpha}\right|_{a=0} \frac{\partial}{\partial x^\mu}$

$x' = x\cos\theta + y\sin\theta$

$y' = -x\sin\theta + y\cos\theta$

$z = z$

$x' = \gamma(x+vt) \quad t' = \gamma(t+vx)$

$$k_x = i\left(\frac{\partial x'}{\partial v}\Big|_{v=0}\frac{\partial}{\partial x} + \frac{\partial t'}{\partial v}\Big|_{v=0}\frac{\partial}{\partial t}\right)$$

$$= i\left(t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t}\right)$$

k_y, k_z $\qquad$ J_i, k_i

$[k_x, k_y] = -J_z$ $\qquad$ $L_i^{int} = -i\epsilon_{ijk}x_j\partial_k$

$[k_x, J_y] = ik_z$ $\qquad$ $J^{mn} = \epsilon_{mni}L_i^{int} =$

$[k_x, J_x] = 0$

$$-i\epsilon_{mni}\epsilon_{ijk}x^j\partial_k$$

$$= -i(x^m\partial^n - x^n\partial^m)$$

$$J^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu)$$

$$= \begin{cases} J_{ij} = -J_{ji} = \epsilon_{ijk}J_k \\ J_{i0} = -J_{0i} = -k_i \end{cases}$$

$$[J_{\mu\nu}, J_{\rho\sigma}] = i(g_{\nu\rho}J_{\mu\sigma} - g_{\mu\rho}J_{\nu\sigma} + g_{\mu\sigma}J_{\nu\rho} - g_{\nu\sigma}J_{\mu\rho})$$

$$[P_\mu, P_\nu] = 0$$

$$[P_\mu, J_{\rho\sigma}] = i(g_{\mu\rho}P_\sigma - g_{\mu\sigma}P_\rho)$$

When $\mu = 0$.

$$P_0 = H$$

$$[P_\nu, H] = 0$$

$$[H, J_{ij}] = i(g_{0i} P_j - g_{0j} P_i) = 0$$

$$[H, J_{i0}] = i(g_{0i} P_0 - g_{00} P_i)$$

$$= -i P_i \neq 0$$

$-K_i$

$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$ $\qquad \Lambda^\mu{}_\rho \Lambda^\sigma{}_\mu = \delta^\sigma_\rho$ $\qquad (\Lambda^{-1})^\mu{}_\nu = \Lambda_\nu{}^\mu$

$\{\bar\Lambda, \bar a\}\{\Lambda, a\} = \{\bar\Lambda\Lambda, \bar\Lambda a + \bar a\}$. $\qquad \{1, 0\}$.

$x''^\mu = \bar\Lambda^\mu{}_\nu (x'^\nu) + \bar a^\mu$ $\qquad \cancel{\bar\Lambda\Lambda,\ a + \bar a}$

$= \underline{\bar\Lambda^\mu{}_\nu \Lambda^\nu{}_k} x^k + \underline{\bar\Lambda^\mu{}_\nu a^\nu + \bar a^\mu}$

$ISO^\uparrow(1,3) \cong \underline{SO^\uparrow(1,3)} \rtimes \mathbb{R}^4$

Wigner method

$P^\mu |p\rangle = p^\mu |p\rangle$, $U(\Lambda, a)|p\rangle = |\Lambda p\rangle$

$P^\mu |\Lambda p\rangle = (\Lambda p)^\mu |\Lambda p\rangle$

$\underline{(\Lambda p)^\mu (\Lambda p)_\mu} = \underline{p^\mu p_\mu = p^2 = m^2}$

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first Casimir invariant.

p^2, p^0 unchanged

$C_1 = P^\mu P_\mu$

six distinct classes

(i) $p^2 = m^2 > 0$, $p^0 > 0$

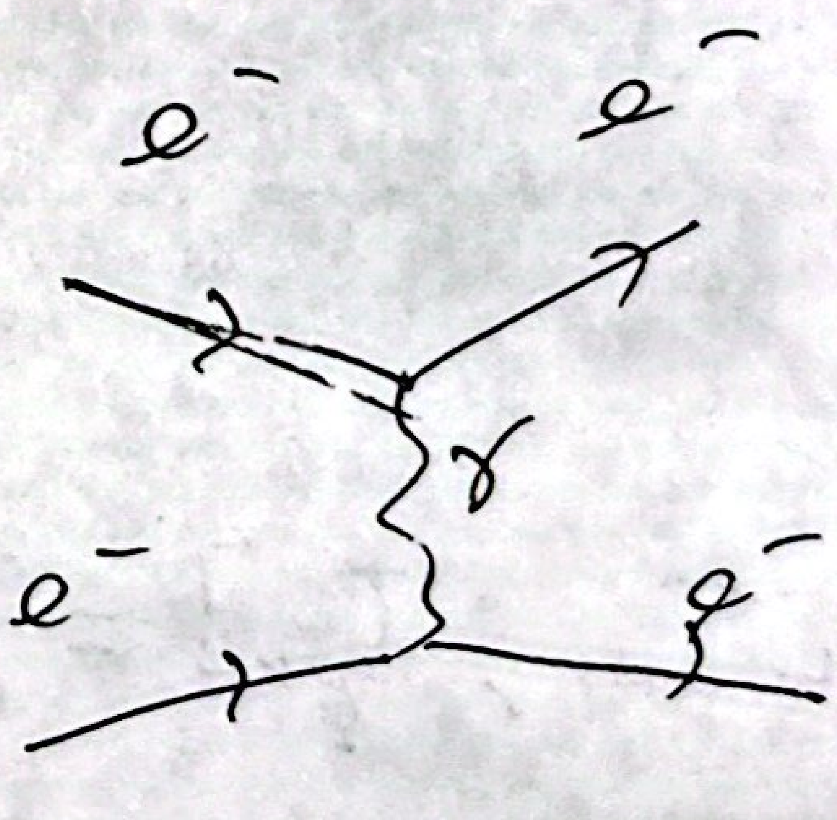
(ii) $p^2 = m^2 > 0$, $p^0 < 0$

(iii) $p^2 = 0$, $p^0 > 0$

(iv) $p^2 = 0$, $p^0 < 0$.

(v) $p^\mu = 0$ $\longrightarrow$ vacuum

(vi) $p^2 < 0$ $\longrightarrow$ virtual particle



$p^\mu \in \{p^\mu\}$

little group of p^μ.

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p^μ at rest frame — k^μ

$k^\mu = (m, 0, 0, 0) \longrightarrow$ rotational group $SU(2)$

$k^\mu \longrightarrow p^\mu$

$p^\mu = L^\mu{}_\nu(p)\, k^\nu$

unitary operator

$|p, \sigma\rangle = U(L(p))\,|k, \sigma\rangle$

$p^\mu \to p'^\mu = \Lambda^\mu{}_\nu\, p^\nu$

$|p, \sigma\rangle \longrightarrow U(\Lambda)\,|p, \sigma\rangle$

$= U(\Lambda)\, U(L(p))\,|k, \sigma\rangle$

$= U(L(\Lambda p))\, U^{-1}(L(\Lambda p))\, U(\Lambda)\, U(L(p))\,|k, \sigma\rangle$

$= U(L(\Lambda p))\, \underline{U(L^{-1}(\Lambda p))\, U(\Lambda)\, U(L(p))}\,|k, \sigma\rangle$

$= U(L(\Lambda p))\, \underbrace{U(L^{-1}(\Lambda p)\, \Lambda\, L(p))}_{\parallel \atop U(R)}\,|k, \sigma\rangle.$

$U(R)\,|k, \sigma\rangle$
$= \sum_{\sigma'} D_{\sigma'\sigma}(R)\,|k, \sigma'\rangle$

$U(\Lambda)\,|p, \sigma\rangle = U(L(\Lambda p)) \sum_{\sigma'} D_{\sigma'\sigma}(R)\,|k, \sigma'\rangle$

$= \sum_{\sigma'} D_{\sigma'\sigma}(R)\, U(L(\Lambda p))\,|k, \sigma'\rangle$

$= \sum D_{\sigma'\sigma}(R)\,|\Lambda p, \sigma'\rangle$

Mass $\rightarrow$ $C_1 = P^\mu P_\mu$

Spin $\rightarrow$? Not J^2

Pauli - Lubanski Pseudo vector

$$W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma} J^{\nu\rho} p^\sigma$$

$$W^\mu P_\mu = 0 \longrightarrow W^\mu \perp p^\mu$$

In rest, $W^\mu = (0, \vec{W})$

$$W_i = -\frac{1}{2}\epsilon_{i\mu\rho\sigma} J^{\mu\rho} p^\sigma$$

$$= -\frac{m}{2}\epsilon_{i\mu\rho 0} J^{\mu\rho}$$

$$= -\frac{m}{2}\epsilon_{ijk0} J^{jk}$$

$$= -m S_i$$

$$\epsilon_{ijk0} J^{jk} = 2J^i = 2S^i$$

$$C_2 = W_\mu W^\mu = -m^2 S_i S_i = -m^2 s(s+1)$$

~~like~~ light - like $p^2 = 0$

$k^\mu = (k, 0, 0, k)$

ISO(2)

$W \cdot W |k\rangle = 0$, $P \cdot P |k\rangle = 0$

$W \cdot P |k\rangle \stackrel{?}{=} 0$

$(W^\mu - \lambda P^\mu)|k\rangle = 0$

$\lambda = W^\mu / p^\mu$ unit: $[J]$

I

Neutrino

$$\vec{\sigma} \cdot \hat{p} \, \phi_L = -\phi_L$$

$$\underline{\frac{1}{2}\vec{\sigma} \cdot \hat{p}} \; \phi_L = \underline{-\frac{1}{2}} \phi_L$$

$$W = \underline{\frac{1}{2} \sigma \cdot p}$$

$$\lambda = -\frac{1}{2}$$

Photon

$$\lambda = \pm 1$$