Group theory

Lorentz group $\longrightarrow$ isomorphic matrix group

$O(1,3)$ $\qquad\qquad\qquad\qquad\qquad\qquad \uparrow$ rep

$SU(2) \cong U(1)$ $\qquad\qquad$ $SO(3) \cong SU(2)/Z_2$

$$g_{\mu\nu} \Lambda^\mu{}_a \Lambda^\nu{}_b = g_{ab}$$

$$(\Lambda^T)_a{}^\mu g_{\mu\nu} \Lambda^\nu{}_b = g_{ab} \qquad \Rightarrow \Lambda^T g \Lambda = g$$

$$g \Lambda^T g = \Lambda^{-1}$$

Spatial

$|\Lambda| = \pm 1$, $J = |\frac{\partial x'}{\partial x}| = |\Lambda| = \pm 1$

proper $\quad |\Lambda| = 1 \qquad SO(1,3)$

improper $\quad |\Lambda| = -1 \longrightarrow P = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

Temporal

$$1 = g_{00} = g_{\mu\nu} \Lambda^\mu{}_0 \Lambda^\nu{}_0 = (\Lambda^0{}_0)^2 - \sum_i (\Lambda^i{}_0)^2$$

$$\Rightarrow (\Lambda^0{}_0)^2 = 1 + \sum_i (\Lambda^i{}_0)^2 \geq 1$$

$$\begin{cases} \Lambda^0{}_0 \geq 1 & \text{orthochronous} \quad SO^\uparrow(1,3) \\ \Lambda^0{}_0 \leq 1 & \text{antichronous} \end{cases}$$

$T = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}$

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

Double cover $\quad SO(3) \cong SU(2)/\mathbb{Z}_2$

$$SO^{\uparrow}(m,n) \xrightarrow[\text{cover}]{\text{double / universal}} \mathrm{Spin}(m,n)$$

Connected spinor group

$$O(m,n) = \{\Lambda \in GL(m+n, \mathbb{F}),\ g\Lambda^T g = \Lambda^{-1})$$

$$g = \begin{pmatrix} \overbrace{+1 \quad +1 \quad +1 \quad \cdots}^{m} & \\ & \overbrace{-1 \quad -1 \quad -1 \quad -1}^{n} \end{pmatrix}$$

$$\mathrm{Spin}(0,1) = \mathbb{Z}_2$$

$$\mathrm{Spin}(0,2) = U(1) = SO(2) = S^1$$

$$\mathrm{Spin}(0,3) = SU(2) = S^3.$$

$\mathrm{Spin}(0,4) =$ $SO(4)$'s double cover $= SU(2) \times SU(2)$.

$$\mathrm{Spin}(1,3) \cong SL(2,\mathbb{C}) \equiv \{ M \in GL(2,\mathbb{C}) \mid |M| = 1 \}.$$

Lie theory.

Group action

generator

$$h(\theta) = \lim_{N \to \infty} \left(1 + \frac{\theta}{N} X\right)^N = e^{\theta X}$$

$$= 1 + \frac{dh}{d\theta} \cdot \theta \cdot \epsilon + \frac{1}{2} \frac{d^2 h}{d\theta^2} \theta^2 \epsilon^2 + \cdots$$

$$= \cancel{e^{\theta X}} \cdot e^{\left.\frac{dh}{d\theta}\right|_{\theta=0} \cdot \theta}$$

$$X = \left.\frac{dh}{d\theta}\right|_{\theta=0}$$

$$[X, Y] = XY - YX.$$

$$e^X e^Y = e^{X + Y + \frac{1}{2}[X,Y] + \cdots} \qquad \text{BCH formula}$$

Formally : 1. Bilinearity

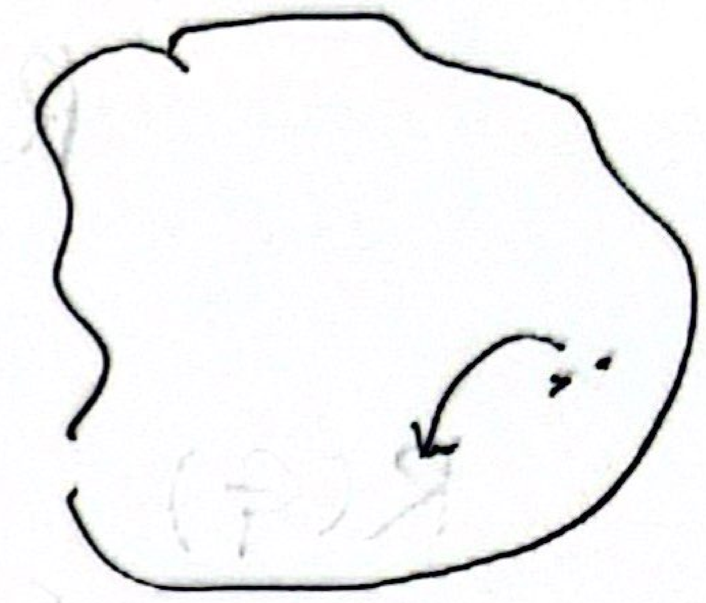
2. Anti commutativity

3. Jacobi identity.

$$[[A,B],C] + [[B,C],A] + [[C,A],B] = 0.$$

Lie group.

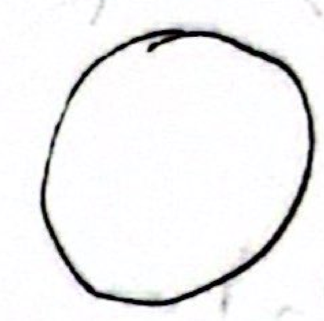
element, point, action, matrix



$U(1),\ SO(2) \cong S^1$

$ab = c.$

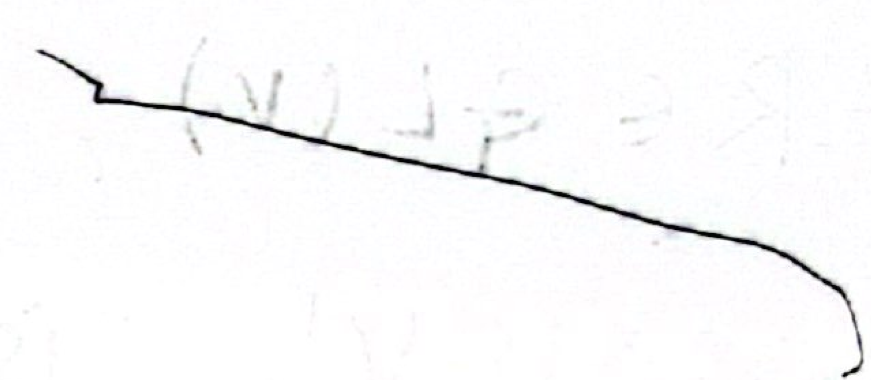
$a^2+b^2=1$



$SU(2) \cong S^3$

$a^2+b^2+c^2+d^2=1$

$SO(3) \cong S^3/Z_2 \cong \mathbb{R}P^3$

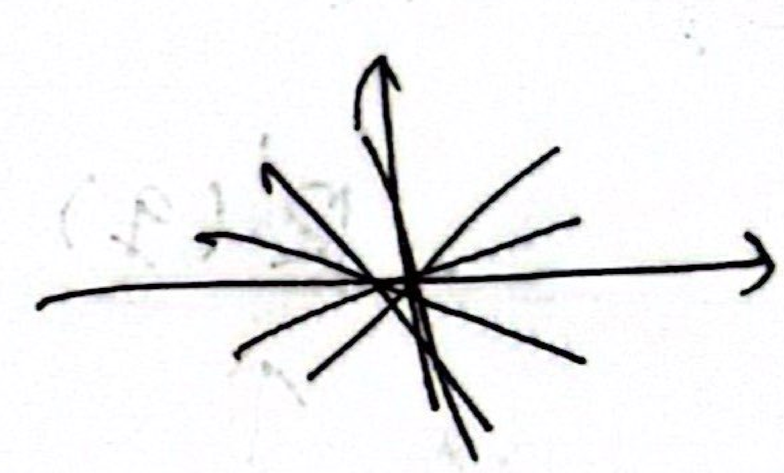


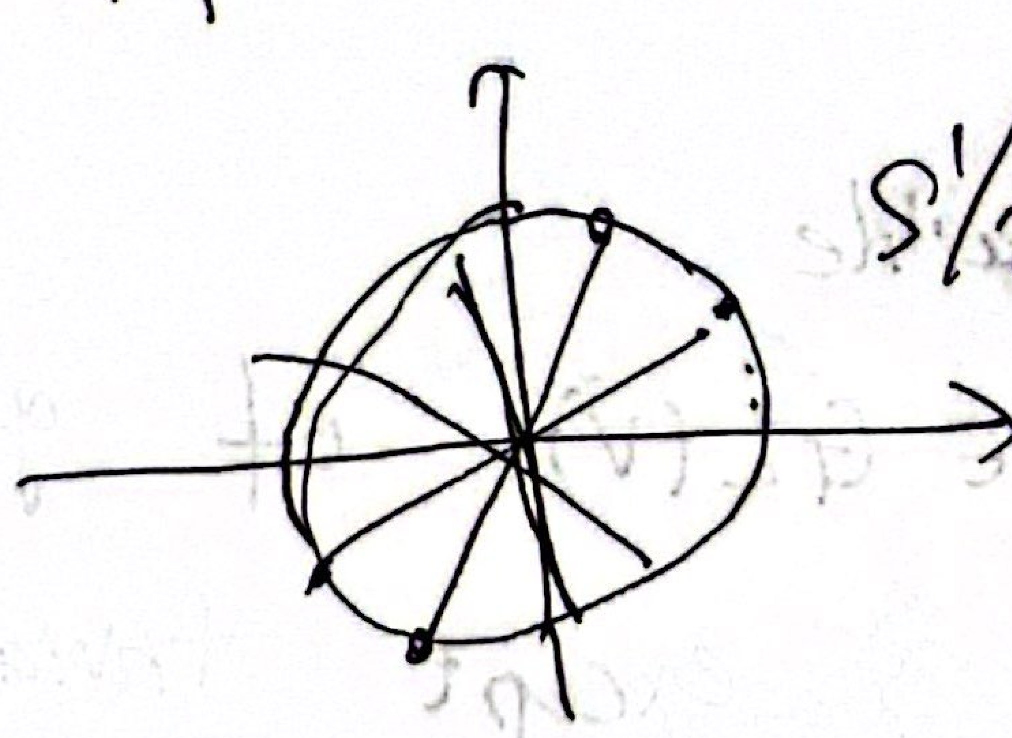
↑

real projection space

$\mathbb{R}^{n+1}$

$S^n/Z_2 \cong \mathbb{R}P^n$





$S^1/Z_2 \cong \mathbb{R}P^1$

$\mathbb{R}P^1$

$SO^\uparrow(1,3) \cong SL(2,\mathbb{C})/Z_2 \cong PSL(2,\mathbb{C})$

$[J_i, J_j] = i\epsilon_{ijk} J_k$

Representation theory

rep.

A rep is a mapping

$$R: G \longrightarrow GL(V), \quad g \mapsto R(g)$$

$$R(e) = I \qquad R(g^{-1}) = (R(g))^{-1} \qquad R(g)\cdot R(h) = R(gh)$$

$\mathbb{R}^3$.

any $R(G)$, any invertible matrix S

$$R \to R' = S^{-1} R S \quad \text{is still a rep of } G.$$

$$R'(g_1) R'(g_2) = S^{-1} R(g_1) S S^{-1} R(g_2) S = S^{-1} R(g_1 g_2) S = R'(g_1 g_2)$$

$\forall R \in GL(V)$, $\exists V' \subset V$, $\forall v \in V'$, $g \in G$, $R(g) v \in V'$

Then V' is invariant subspace.

def. R'

$$R'(g) v = R(g) v, \quad \forall v \in V', g \in G.$$

Then R is reducible

An irreducible rep $R \in GL(V)$ of group G means ~~was~~, no invariant subspace, except trivial 0, V.

e.g. $|s, m_s\rangle$.

$$S_x \to \langle s', m_s' | S_x | s, m_s \rangle = \begin{pmatrix} (0) & & \\ & \begin{pmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{pmatrix} & \\ & & \begin{pmatrix} 0 & \hbar/\sqrt{2} & 0 \\ \hbar/\sqrt{2} & 0 & \hbar/\sqrt{2} \\ \end{pmatrix} \end{pmatrix}$$

$$S_z = \begin{pmatrix} 0 & & \\ & \underbrace{\begin{pmatrix} \hbar/2 & 0 \\ 0 & \hbar/2 \end{pmatrix}}_{1/2} & \\ & & \underbrace{\begin{pmatrix} \hbar & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\hbar \end{pmatrix}}_{1} \end{pmatrix}$$

rep of Lie algebra

$$\pi : \mathfrak{g} \longrightarrow \mathfrak{gl}(n, \mathbb{C})$$

$$\forall X, Y \in \mathfrak{g} : \quad \pi([X, Y]) = [\pi(X), \pi(Y)].$$

π is ~~irreducible~~ reducible, if $\exists\, 0 \neq W \subset V$,

$$\forall X \in \mathfrak{g}, \quad \pi(X)\, w \subseteq W.$$

How to find fundamental reps.

Casimir element, $\quad [C, X] = 0$ ← generator

Schur's lemma.

$SU(2)$, $\quad [J^2, J_i] = 0 \qquad J^2 = \underline{j(j+1)\hbar^2}.$

Poincaré group $\quad P^2, W^2$

<u>Cartan subalgebra</u>

field is a map from manifold M to manifold V (vector space

$$\phi : M \to V$$

$(\mathbb{R}^4, g)$.

Scalar trivial

spinor $(0, 1/2) \oplus (1/2, 0)$

$SO(1,3,\mathbb{C}) = SU(2) \otimes SU(2)$

$\hookrightarrow SL(2,\mathbb{C})$

$\phi^i \phi_i$

$|x\rangle \quad |p\rangle$

V

$\langle E_n | E_m \rangle = \delta_{nm}$

Poincare group $\quad ISO^{\uparrow}(1,3) \cong SO^{\uparrow}(1,3) \rtimes \mathbb{R}^4$.

$\mathbb{R}^4 \triangleleft ISO^{\uparrow}(1,3)$

$SO^{\uparrow}(1,3) \ntriangleleft ISO^{\uparrow}(1,3)$

$\forall g \in G,\ n \in N,$ iff $gng^{-1} \in N,$

$\cancel{U(\Lambda_2, a_2)\Lambda(}$

$U(\Lambda_2, a_2)\, U(\Lambda_1, a_1) = U(\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2)$

$U^{-1}(\Lambda, a) = U(\Lambda^{-1}, -\Lambda^{-1}a)$.

$U^{-1}(\Lambda, a)\, U(1, a')\, U(\Lambda, a)$

$J^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu)$

$= U(\Lambda^{-1}, -\Lambda^{-1}a)\, U(\Lambda, a + a')$

$= U(1, \Lambda^{-1}a') \in \mathbb{R}^4$

$U^{-1}(\Lambda, a)\, U(\Lambda', 0)\, U(\Lambda, a)$

$= U(\Lambda^{-1}, -\Lambda^{-1}a)\, U(\Lambda'\Lambda, \Lambda' a)$

$= U(\Lambda^{-1}\Lambda'\Lambda, (\Lambda^{-1}\Lambda' - \Lambda^{-1})a) \notin SO^{\uparrow}(1,3)$

reps of Lorentz group. $\quad N_i^+ \quad N_i^-$

(j_1, j_2)

1. $(0,0)$ $\quad N_i^+ = N_i^- = 0$

$$e^{N_i^+} = e^{N_i^-} = 1$$

$$\Lambda = \exp(-i\theta J - i\phi K) = 1$$

$\phi(x)$

$$\phi \to \Lambda\phi = \phi$$

$$\phi'(\Lambda x) = \phi(x)$$

$$\phi'(x) = \phi(\Lambda^{-1}x)$$

R_1 $\quad \phi_i'(x) = (R_1(\Lambda))_i{}^j \phi_j(\Lambda^{-1}x)$

$$\phi_i'(x)\psi_a'(x) = (R_1(\Lambda))_i{}^j (R_2(\Lambda))_a{}^b \phi_j(\Lambda^{-1}x)\phi_b(\Lambda^{-1}x)$$

$$R_1 \otimes R_2$$

$$j_1 \otimes j_2 = (j_1+j_2) \oplus (j_1+j_2-1) \oplus \cdots \oplus |j_1-j_2|$$

$$= \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} j$$

$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$

$|\uparrow\rangle \quad |\downarrow\rangle$

$|\uparrow\uparrow\rangle \quad |\downarrow\downarrow\rangle$ $\quad \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$

$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$

$$(j_1, j_1') \otimes (j_2, j_2')$$

$$= (j_1 \otimes j_2)_{R_1} \otimes (j_1' \otimes j_2')_{R_2}$$

$$= ((j_1+j_2) \oplus \cdots \oplus |j_1-j_2|) \otimes ((j_1'+j_2') \oplus \cdots \oplus |j_1'-j_2'|)$$

$$= \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} \bigoplus_{j'=|j_1'-j_2'|}^{j_1'+j_2'} (j, j')$$

$j_1 = j_2, \quad j_1' = j_2'$

$p^{ia}\psi_i(x)\phi_a(x)$

Projector

$$p^{ia} \phi_i'(x) \psi_a'(x) = p^{jb} \phi_j(\Lambda^{-1}x) \psi_b(\Lambda^{-1}x)$$

$$= p^{ia} R(\Lambda)_i{}^j \phi_j(\Lambda^{-1}x) R(\Lambda)_a{}^b \psi_b(\Lambda^{-1}x)$$

$$\Rightarrow \quad p^{ia} R(\Lambda)_i{}^j R(\Lambda)_a{}^b = p^{jb}.$$

$$R(\Lambda)_i{}^j = (1 + w_{\mu\nu} J^{\mu\nu})_i{}^j$$

$$= \delta_i{}^j + w_{\mu\nu} (J^{\mu\nu})_i{}^j$$

$$p^{ia} \left(\delta_i{}^j \delta_a{}^b + w_{\mu\nu} \left[(J^{\mu\nu})_i{}^j \delta_a^b + (J^{\mu\nu})_a{}^b \delta_i^j \right] \right) = p^{jb}$$

$$p^{ib} R(J^{\mu\nu})_i{}^j + p^{ja} R(J^{\mu\nu})_a{}^b = 0.$$

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2. $(1/2, 0)$ $\quad N_i^- = 0 \quad , \quad N_i^+ = \frac{\sigma_i}{2}$

$$J_i = i K_i \quad , \quad i K_i = \frac{\sigma_i}{2}$$

$$\Rightarrow \quad S^{0i} = -\frac{i}{2} \sigma^i \quad , \quad S^{ij} = \epsilon^{ijk} J^k = \frac{1}{2} \epsilon^{ijk} \sigma^k.$$

$SL(2, \mathbb{C})$

$$p^{ja} (\sigma^k)_a{}^b + p^{ib} (\sigma^k)_i{}^j = 0.$$

$$p^{ij} = \epsilon^{ij} \qquad \epsilon^{ij} \phi_i \psi_j$$

$$\epsilon^{ij} \phi_i \psi_j$$

$$\phi_1 \psi_2 - \phi_2 \psi_1 \qquad \underset{4}{\phi_i \phi_j} \longrightarrow \underset{3}{\text{symmetric}} + \underset{1}{\text{anti}}\text{--}$$

$X'^{\mu} = \Lambda^{\mu}{}_{\nu} X^{\nu}$ $\quad X'_{\mu} = \Lambda_{\mu}{}^{\nu} X_{\nu} = (\Lambda^{-1})^{\nu}{}_{\mu} X_{\nu}$

$= (\Lambda^{-1})^{T}{}_{\mu}{}^{\nu} X_{\nu}$

$\psi_a \to R_a{}^b \psi_b$ fundamental

$\psi^a \to (R^{-1})^{T a}{}_b \psi^b$ dual

$\bar{\psi}_{\dot{a}} \to R^{*}{}_{\dot{a}}{}^{\dot{b}} \bar{\psi}_{\dot{b}}$ conjugate

$\bar{\psi}^{\dot{a}} \to (R^{-1})^{\dagger \dot{a}}{}_{\dot{b}} \bar{\psi}^{\dot{b}}$ conjugate dual.

$$\psi'_a = \psi_a + i\theta^i (G_R^i)_a{}^b \psi_b.$$

$$\bar{\psi}'_{\dot{a}} = \psi_a^* - i\theta^i (G_R^i)_a^{*}{}^b \psi_b^*$$

$$\Rightarrow \quad G_{\bar{R}}^i = -(G_R^i)^*$$

$$U_{\bar{R}} = \exp(i\theta^i G_{\bar{R}}^i) = \exp(-i\theta^i (G_R^i)^*) = U_R^*.$$

3. $(0, 1/2)$

$$S^{0i} = \frac{i}{2}\sigma^i, \quad S^{ij} = \frac{1}{2}\epsilon^{ijk}\sigma^k$$

$$-\sigma_i^* = \sigma_2 \sigma_i \sigma_2$$

$$\underbrace{-(\sigma_i^{*})_a{}^d}_{S_R{}^d{}_a} = (-i\sigma_2)_a{}^b \underline{(\sigma_i)_b{}^c} (i\sigma_2)_c{}^d.$$

$$= (i\sigma_2)^{-1}{}_a{}^b \underbrace{(\sigma_i)_b{}^c}_{S_L} (i\sigma_2)_c{}^d$$

$$\psi_R = i\sigma_2 \psi_L^*.$$

$\epsilon^{ab} \psi_{La} \phi_{Lb}$ $\quad$ $\epsilon^{AB} \psi_{RA} \phi_{RB}$.

$\epsilon^{AB} \psi_{RA} (i\sigma_2 \phi_L^*)^*_B$ $\quad$ $\epsilon^{ab} \psi_{La} (-i\sigma_2 \phi_R^*)_b$

$$\psi_L^a = \epsilon^{ab} \psi_{Lb}$$
$$\psi_R^A = -\epsilon^{AB} \psi_{LB}.$$

$j_1 + j_2 = j$.

4. $(1/2, 1/2)$ Vector rep of Lorentz group.

$(1/2, 1/2) = (0, 1/2) \otimes (1/2, 0)$ $\quad$ $\exp(-i\omega_{\mu\nu} J^{\mu\nu})_a{}^b = \Lambda = \begin{pmatrix} \gamma & -\gamma\beta & \\ -\gamma\beta & \gamma & \\ & & \ddots \end{pmatrix}$

$\sigma^\mu \equiv \sigma^\mu_{a\dot{a}}\, e^a \otimes e^{\dot{a}}$

$\cancel{(J^{\mu\nu}) = i(g^{a\mu}\delta^b{}_\nu}$

$$\overset{4}{(1/2, 1/2)} \otimes \overset{4}{(1/2, 1/2)}$$

$(J^{ab})^\mu{}_\nu = i(g^{a\mu}\delta^b{}_\nu - g^{b\mu}\delta^a{}_\nu)$

$$= \underbrace{(0,0)}_{1} \oplus \underbrace{(0,1) \oplus (1,0)}_{6 \leftarrow \text{anti}} \oplus \underbrace{(1,1)}_{9}.$$

$10 \leftarrow$ symmetric

J_i, K_i $\quad$ $g^{\mu\nu}$

$g^{\mu\nu} A_\mu B_\nu = A_\mu B^\mu$, $(\partial_\mu A^\mu)$

$$(0, 1/2) \otimes (1/2, 0) \otimes (1/2, 1/2)$$

$\sigma^\mu_{a\dot{a}} \phi^a \bar{\phi}^{\dot{a}} A_\mu$ $\quad$ $\sigma^\mu_{a\dot{a}} \phi^a \partial_\mu \bar{\phi}^{\dot{a}}$

5. Tensor $\quad$ $(A_\mu B^\mu)^2$ $\quad$ $(\partial_\mu A^\mu)^2$.

$$\epsilon^{\mu\nu\rho\sigma} A_\mu B_\nu C_\rho D_\sigma$$

$$\epsilon^{\mu\nu\rho\sigma} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\rho A_\sigma - \partial_\sigma A_\rho).$$

$$[(0,1) \oplus (1,0)] \otimes [(0,1) \oplus (1,0)] \to (0,0)$$

- Dirac spinor $(0, 1/2) \oplus (1/2, 0)$

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] \,.$$

$$[(0, 1/2)_L \oplus (\tfrac{1}{2}, 0)_R] \otimes [(0, 1/2)_L \oplus (1/2, 0)_R]$$

$$= \underbrace{(0,0)_{LL} \oplus (0,0)_{RR}}_{\bar{\Psi}\Psi,\ \bar{\Psi}\gamma^5\Psi} \oplus \underbrace{(\tfrac{1}{2}, \tfrac{1}{2})_{LR} \oplus (\tfrac{1}{2}, \tfrac{1}{2})_{RL}}_{\bar{\Psi}\gamma^\mu\Psi A_\mu,\ \partial_\mu \bar{\Psi}\gamma^\mu\gamma^5\Psi} \oplus \underbrace{(0,1)_{LL} \oplus (1,0)_{RR}}_{\bar{\Psi}\sigma^{\mu\nu}\Psi}$$

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

$\bar{\Psi}\Psi$ $\qquad$ $\bar{\Psi}\gamma^\mu\Psi \underline{A_\mu}$ $\qquad$ $\underline{\bar{\Psi}\sigma^{\mu\nu}\Psi}$

$\bar{\Psi}\gamma^5\Psi$ $\qquad$ $\partial_\mu \bar{\Psi}\gamma^\mu\gamma^5\Psi \cancel{A_\mu}$ $\qquad$ $\bar{\Psi}\sigma^{\mu\nu}\Psi [A_\mu B_\nu - A_\nu B_\mu]$

$$\bar{\Psi} = \Psi^\dagger \gamma_0$$

- $(1,1)$ graviton spin 2

$(1/2, 1) \oplus (1, 1/2)$ Ravita - Schwinger field 3/2 fermions.

12.

$(0,1) \oplus (1,0)$ $\qquad$ $F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$\phi_i(x)$

$$J^{ij} = L^{ij} + S^{ij}$$

$$= -\int d^3x\, \pi^b \underline{(x^i\partial_j - x^j\partial_i)} \phi_b(x)$$

$$- i \int d^3x\, \pi^b (J^{ij})_b{}^a\, \phi_a(x)$$