讨论班 2.8& 2.9

Maxwell Equations and Proca Equations.

1) Maxwell equations.

$$\begin{cases} \nabla\cdot B = 0 & ① \\ \nabla\times B = j + \frac{\partial E}{\partial t} & ② \\ \nabla\times E = -\frac{\partial B}{\partial t} & ③ \\ \nabla\cdot E = \rho & ④ \end{cases}$$

Vector form $\xrightarrow{SR}$ Covariant form.

$\downarrow$

Lorentz trans.

introduce : Magnetic vector potential : A

$$\begin{cases} B = \nabla\times A \\ E = -\frac{\partial A}{\partial t} - \nabla\phi \end{cases}$$

4-Vector $A^\mu = (\phi, \vec{A})$

$$\begin{cases} E^i = \partial^0 A^i - \partial^i A^0 \\ B^i = \varepsilon^{ijk}\partial^j A^k \end{cases}$$

EMF Tensor: $F^{\mu\nu} = -F^{\nu\mu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}$

1). Anti-sys.

2). Traceless

under Lorentz transformation:

$$F^{\mu\nu} \to \Lambda^\mu_{\ \alpha}\Lambda^\nu_{\ \beta}F^{\alpha\beta}$$

2) Gauge

$$\begin{aligned} A &\to A - \nabla\chi \\ \phi &\to \phi + \frac{\partial\chi}{\partial t} \end{aligned} \Rightarrow A^\mu \to A^\mu + \partial^\mu\chi$$

$\triangle$

$$F'^{\mu\nu} = F^{\mu\nu} + (\partial^\mu\partial^\nu - \partial^\nu\partial^\mu)\chi = F^{\mu\nu}$$

① $\Box A^\mu = j^\mu + \partial^\nu(\partial_\nu A^\mu)$

$\downarrow$

0

Lorentz gauge

$$\partial_\mu A^\mu = \frac{\partial\phi}{\partial t} + \nabla\cdot A = 0$$

$\Box A^\mu = j^\mu$ ⓪ For vanc

$$\begin{cases} \frac{\partial^2 \phi}{\partial t^2} - \nabla^2\phi = \rho \\ \frac{\partial^2 A}{\partial t^2} - \nabla^2 A = \vec{j} \end{cases}$$

(Liénard - Wiechert potential.)

$\Box A^\mu = 0$

Back to Maxwell equation:

$$\partial_\mu F^{\mu\nu} = j^\nu \quad ☆$$

$$\Rightarrow \begin{cases} \nu = 0 & : \nabla\cdot E = \rho \quad ② \\ \nu = 1,2,3, & \nabla\times B = j + \frac{\partial E}{\partial t} \quad ④ \end{cases}$$

$$F = -\frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu$$

$$*F^{\mu\nu} = G^{\mu\nu} = \frac{1}{2}\underbrace{\varepsilon^{\mu\nu\rho\sigma}}_{\text{4-D Levi-Civita}} F_{\rho\sigma} = \begin{pmatrix} 0 & -B^1 & -B^2 & -B^3 \\ B^1 & 0 & E^3 & -E^2 \\ B^2 & -E^3 & 0 & E^1 \\ B^3 & E^2 & -E^1 & 0 \end{pmatrix}$$

$$\partial_\mu G^{\mu\nu} = 0 \Rightarrow ①③$$

Maxwell equation → Tensor form.

$$\begin{cases} \partial_\mu F^{\mu\nu} = J^\nu \\ \partial_\mu G^{\mu\nu} = 0 \end{cases}$$

3) Proca Equations: For massive spin-1 particles.

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$\partial_\mu F^{\mu\nu} + \underbrace{m^2}_{\text{mass}} A^\nu = 0$$

$$m^2 \partial_\nu A^\nu = 0 \qquad (\Box + m^2) A^\mu = 0 \qquad (\partial_\mu A^\mu = 0)$$

Proca Equation $\begin{cases} (\Box + m^2) A^\mu = 0 \\ \partial_\mu F^{\mu\nu} + m^2 A^\nu = 0 \end{cases}$

2.9. Differential Form.

Intro:

$$\begin{cases} I_1 = \int_C F_x dx + F_y dy + F_z dz = \int_C F \cdot dr = \mathbb{R} \\ I_2 = \int_S (G_x dy \wedge dz + G_y dz \wedge dx + G_z dx \wedge dy) \\ \quad = \int_S G \cdot ds = \mathbb{R} \end{cases}$$

orientation

$C \leftrightarrow S$

$\uparrow$ dual $\uparrow$

1. $C_n \to$ n-D volume
2. Boundary Operator : ∂ $\quad C_n \xrightarrow{\partial} C_{n-1} \quad \partial C_n \to C_{n-1}$
3. $\partial Z_n = 0 \quad Z_n$. no boundary
4. $B_n = \partial C_{n+1} \Rightarrow \partial B_n = 0$

   $\partial^2 = 0$

   $\to Z_n$ $\quad$ for Euclidean $Z_n = B_n$, $\quad Z_n \supset B_n$

5. w_0 0-form $\to$ function $\subset$ 3-D

   w_1 1-form $\to A\,dx + B\,dy + C\,dz$

   w_2 2-form $\to f\,dx \wedge dy + g\,dy \wedge dz + h\,dz \wedge dx$

   w_3 3-form $\to F\,dx \wedge dy \wedge dz$

$dx \wedge dy = -dy \wedge dx \quad , \quad dx \wedge dx = 0$

6. Exterior derivative operator:

$$dw_n = w_{n+1}$$

3-D

$$d(A\,dx + B\,dy + C\,dz) = \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dx \wedge dy + \left(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z}\right) dy \wedge dz + \left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}\right) dz \wedge dx$$

$$d(A, B, C) \to \nabla \times (A, B, C)$$

$$dw_2 = \left(\frac{\partial f}{\partial z} + \frac{\partial g}{\partial x} + \frac{\partial h}{\partial y}\right) dx \wedge dy \wedge dz$$

$$\nabla \cdot (g, h, f)$$

$$d^2 = 0 \quad (\nabla \cdot (\nabla \times \vec{v}) = 0)$$

$$\begin{cases} d\omega_n = 0 \Rightarrow \omega_n \text{ is closed} \\ \omega_n = d\omega_{n-1} \Rightarrow \omega_n \text{ is exact} \end{cases}$$

Stokes : $\int_{\partial C} \omega = \int_C d\omega$

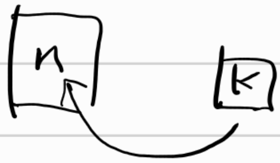
4) Hodge dual stor operator

For k-d Vector space (inner product).

$\langle \alpha, \beta \rangle = \det\left(\langle \alpha_i, \beta_j \rangle_{i,j=1}^{k} \right)$ $\alpha, \beta \in \Lambda^k V$ $(0 \leq k \leq n)$

$\{e_1, e_2, \ldots, e_n\}$ of V as :

$\omega := e_1 \wedge e_2 \wedge \ldots \wedge e_n$



Hodge : $\alpha \wedge (* \beta) = \langle \alpha, \beta \rangle \omega$

$K\text{-}D \rightarrow (n-k)\text{-}D$

$\beta \rightarrow * \beta$

$*(w_1 \wedge \ldots \wedge w_k) = u_1 \wedge \ldots \wedge u_{n-k}$

Eample :

2-D
$$\begin{cases} *1 = dx \wedge dy \\ * dx = dy \\ * dy = -dx \\ *(dx \wedge dy) = 1 \end{cases}$$

4-D $*(dx^\mu) = \eta^{\mu\lambda} \varepsilon_{\lambda\nu\rho\sigma} \frac{1}{3!} dx^\nu \wedge dx^\rho \wedge dx^\sigma$

$*(dx^\mu \wedge dx^\nu) = \eta^{\mu\kappa} \eta^{\nu\lambda} \varepsilon_{\kappa\lambda\rho\sigma} \frac{1}{2!} dx^\rho \wedge dx^\sigma$

$$F = -\frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$$

$$*F = -\frac{1}{2} F_{\mu\nu} *(dx^\mu \wedge dx^\nu)$$

$$\begin{cases} d*F = J \\ dF = 0 \end{cases}$$

$J = (j_x\, dy \wedge dz + j_y\, dz \wedge dx + j_z\, dx \wedge dy) \wedge dt$
$- \rho\, dx \wedge dy \wedge dz$