Canonical Quantization of EM field

A^μ

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \qquad \partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} = 0$$

$$\partial_\mu \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} = 0$$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$\partial_\mu F^{\mu\nu} = 0 \longrightarrow \Box A^\nu - \cancel{\partial^\nu (\partial_\mu A^\mu)} = 0$$

$$\Box A^\nu = 0$$

$$A_\mu \to A_\mu' + \partial_\mu \Lambda(x)$$

$$\Box \Lambda = -\partial_\mu A^\mu \rightsquigarrow \underline{\partial_\mu A^{\mu'}} = \partial_\mu A^\mu - \partial_\mu A^\mu = 0$$

Lorentz gauge.

$$\partial_\mu A^{\mu'} = \partial_\mu A^{\mu'} + \underline{\Box \Lambda(x)}$$

$$\Box \Lambda(x) = 0$$

$$\phi' = \phi + \partial_0 \Lambda = 0$$

$$\begin{cases} \dfrac{\partial \Lambda}{\partial t} = -\phi \\ \partial_0 A^0 - \nabla \cdot A = 0 \end{cases} \Rightarrow \begin{cases} \phi' = 0 \\ \nabla \cdot \vec{A} = 0 \end{cases}$$

$\cancel{\partial_0 \phi'}$

Coulomb gauge (Radiation).

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

$$= -\frac{1}{2} (\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2} (\partial_\mu A_\nu)(\partial^\nu A^\mu)$$

$$\pi^0 = \frac{\partial L}{\partial \dot{A}_0} = 0$$

$$\pi^i = \frac{\partial L}{\partial \dot{A}_i}$$

$$-\frac{1}{2}\dot{A}_i^2 - \frac{1}{2}\dot{A}_i \partial_i A_0 ,$$

$$-\dot{A}_i + \partial_i A_0 = E_i$$

$$[A_i(x,t), \pi^j(x',t)] = i\,\delta_i{}^j\,\delta^3(\vec{x}-\vec{x}')$$

$$[\vec{\nabla}\cdot\vec{A}(x,t), \pi^j(x',t)] = i\,\partial^i\,\delta^3(\vec{x}-\vec{x}') \neq 0.$$

$$[A^i, \pi^j] = -i\,\Delta^{ij}\int\frac{d^3k}{(2\pi)^3}\,e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}$$

$$[\nabla\cdot A, \pi^j] = \frac{1}{(2\pi)^3}\int d^3k \;\; \underline{k_i\,\Delta^{ij}}\,e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}$$

$$k^i\Delta^{ij} = 0$$

$$\Delta^{ij} = \delta^{ij} - \frac{k^i k^j}{k^2}$$

$$[A^i, E^j] = i\int\frac{d^3k}{(2\pi)^3}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}$$

$$= i\left(\delta^{ij} - \frac{\partial^i\partial^j}{\nabla^2}\right)\delta^3(\vec{x}-\vec{x}')$$

$$[A^i, A^j] = [E^i, E^j] = 0.$$

$$\vec{A}(x) = \int \frac{d^3k}{(2\pi)^3 2k_0} \sum_{\lambda=1}^{2} \vec{\epsilon}^{(\lambda)}(k) \left[a^{(\lambda)}(k) e^{-ikx} + a^{(\lambda)\dagger} e^{ikx} \right] \}.$$

$k^2 = 0$, $k_0 = |k|$,

$\nabla \cdot \vec{A} = 0$

$\Rightarrow \vec{k} \cdot \vec{\epsilon}^{(\lambda)}(\vec{k}) = 0$ $\qquad \epsilon$. polarization vector

$\epsilon^{(\lambda)}(k) \cdot \epsilon^{(\lambda')}(k) = \delta_{\lambda\lambda'}$

$[a^{(\lambda)}(k), a^{(\lambda')\dagger}(k')] = 2k_0 (2\pi)^3 \delta_{\lambda\lambda'} \delta^3(\vec{k} - \vec{k'})$.

$$H = \frac{1}{2} \int d^3x \, (E^2 + B^2)$$

$$= \frac{1}{2} \int d^3x \, (\dot{A}^2 + (\nabla \times A)^2).$$

$$(\nabla \times \vec{A})^2 = \epsilon_{ijk} \epsilon_{imn} \partial_j A_k \partial_m A_n$$

$$= (\partial_j A_k)(\partial_j A_k) - (\partial_j A_k)(\partial_k A_j)$$

$$= \partial_j [A_k \partial_j A_k] - \partial_j [A_k \partial_k A_j]$$

$$\underline{- A_k \nabla^2 A_k} + \cancel{A_k \partial_k (\nabla \cdot A)}$$

$$H = \frac{1}{2} \int (\dot{A}^2 - A \cdot \nabla^2 A) \, d^3x$$

$$= \sum_{\lambda} \int \frac{d^3k}{(2\pi)^3 k_0} \frac{k_0}{2} [a^{(\lambda)\dagger}(k) a^{(\lambda)}(k)]$$

$$\underset{(x,t)}{[A_\mu}, \underset{(x',t)}{\pi_\nu]} = i\, g_{\mu\nu}\, \delta^3(\vec{x}-\vec{x}')$$

$$[A_\mu, A_\nu] = [\pi_\mu, \pi_\nu] = 0$$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2$$

Feynman gauge,

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0.$$

$$[A_0, \pi_0] = 0$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\partial^\mu A^\nu + \partial^\nu A^\mu - g^{\mu\nu}\partial_\lambda A^\lambda$$

$$\frac{\partial \mathcal{L}}{\partial A^\mu} = 0.$$

$$\Box A_\mu = 0.$$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2$$

$$\Box A^\mu - \left(1-\frac{1}{\xi}\right)\partial^\mu(\partial_\lambda A^\lambda) = 0$$ R_ξ gauge

$$\pi^0 = -\partial_\mu A^\mu = 0$$

$$\underline{\langle \psi | \partial_\mu A^\mu | \psi \rangle = 0.}$$

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3 2k_0} \sum_{\lambda=0}^{3} \epsilon_\mu^{(\lambda)}(k)\left(a^{(\lambda)} e^{-ikx} + a^{(\lambda)\dagger}(k)\, e^{ikx}\right).$$

$$\epsilon^{(\lambda)}\epsilon^{(\lambda')} = g^{\lambda\lambda'}$$

timelike

$$\epsilon^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \epsilon^{(1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad e^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$e^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad k^\mu = (k, 0, 0, k)$$

longitudinal

$$k \cdot \epsilon^{(1,2)} = 0$$

unphysical

$$\Pi^\mu = F^{\mu 0} - g^{\mu 0}(\partial_\nu A^\nu)$$

$$\Pi^0 = -\dot{A}^0 - \nabla \cdot A \qquad \Pi^i = \partial^i A^0 - \dot{A}^i$$

$$[A_\mu, \pi_\nu] = i g_{\mu\nu} \delta^3(\vec{x} - \vec{x}')$$

$$[a^{(\lambda)}(k), a^{(\lambda')\dagger}(k')] = -g^{\lambda\lambda'} 2k_0 (2\pi)^3 \delta^3(\vec{k} - \vec{k}')$$

$$[a^{(0)}(k), a^{(0)\dagger}(k')] = -2k_0 (2\pi)^3 \delta^3(k - k').$$

$$|1\rangle = \int \frac{d^3k}{(2\pi)^3 2k_0} f(k)\, a^{(0)\dagger}(k) |0\rangle.$$

$$\langle 1|1\rangle = \cdots \langle 0|[a^{(0)}(k'), a^{(0)\dagger}(k)]|0\rangle$$

$$= -\int \cdots \langle 0|0\rangle$$

$$\langle n_t|n_t\rangle = (-1)^{n_t}$$

$$H = \int \frac{d^3k}{(2\pi)^3 (2k_0)} k_0 \left[\sum_{\lambda=1}^{3} a^{(\lambda)\dagger} a^{(\lambda)} - a^{(0)\dagger} a^{(0)} \right]$$

$$N = -a^\dagger a$$

$$\langle 1|H|1\rangle = -\sum_{\lambda}\int \frac{d^3k}{(2\pi)^3 2k_0}\, k_0\, |f(k)|^2 \langle 0|0\rangle.$$

Gupta – Bleuler formalism

$\cancel{\partial^\mu A_\mu}$ $\quad \cancel{\partial_\mu A^\mu|\psi\rangle = 0}$

$$\partial_\mu A^\mu|\psi\rangle = (\partial_\mu A^{+\mu} + \partial_\mu A^{-\mu})|\psi\rangle = 0.$$

$$(\partial_\mu A^{+\mu})|\psi\rangle = 0$$

$$\langle\psi|\partial_\mu A^\mu|\psi\rangle = \langle\psi|\partial_\mu A^{+\mu} + \partial_\mu A^{-\mu}|\psi\rangle.$$

$$= \langle\psi|\partial_\mu A^{-\mu}|\psi\rangle.$$

$$\sum_{\lambda=0}^{3} k^\mu \epsilon_\mu^{(\lambda)} a^{(\lambda)}(k)|\psi\rangle = 0 \qquad = \langle\psi|\partial_\mu A^{+\mu}|\psi\rangle^*.$$

$$= 0.$$

$$= [k^\mu \epsilon_\mu^{(0)} a^{(0)}(k) + k^\mu \epsilon_\mu^{(3)} a^{(3)}(k)]|\psi\rangle = 0.$$

$$k^\mu \epsilon_\mu^{(0)} = -k^\mu \epsilon_\mu^{(3)}.$$

$$\Rightarrow [a^{(0)}(k) - a^{(3)}(k)]|\psi\rangle = 0$$

$$\langle\psi|a^{(0)\dagger}(k)\, a^{(0)}(k)|\psi\rangle = \langle\psi|a^{(3)\dagger}(k)\, a^{(3)}(k)|\psi\rangle$$

$$\langle O \rangle = \langle \phi | O | \phi \rangle$$

$$= \langle \phi | \eta \, O | \phi \rangle \qquad \eta = (-1)^{n_t}$$

BV – BRST formalism