3.1. Lagrangian formulation of particle mechanics.

$$L = T - V = \frac{1}{2} m \dot{x}^2 - V(x)$$

$$S = \int_{t_1}^{t_2} L\,dt \quad \rightarrow \quad \delta S = 0 = \int_{t_1}^{t_2} (m\dot{x}\dot{\varepsilon} - \varepsilon V'(x))\,dt$$

$$\int m\dot{x}\dot{\varepsilon}\,dt = \dot{x}a\Big|_{t_1}^{t_2} - \int a\ddot{x}\,dt$$

$$\Rightarrow m\ddot{x} = -V'(x)$$

3.2 Scalar field.

$$\mathcal{L}(\phi, \partial_\mu \phi)$$

$$S = \int \mathcal{L}(\phi, \partial_\mu \phi)\, d^4x \quad \Leftrightarrow \quad (\Box + m^2)\phi = 0$$

density

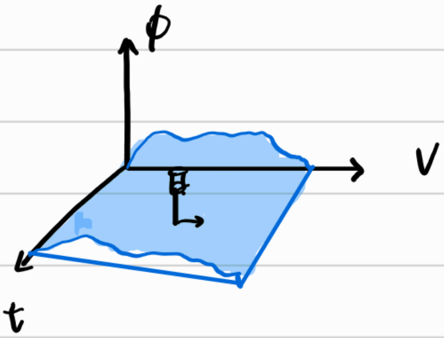
particle

$$L = \int \mathcal{L}(\phi, \partial_\mu \phi)\, d^3x$$

Space

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2$$
$$= \frac{1}{2}\left[(\partial_0 \phi)^2 - (\nabla \phi)^2 - m^2 \phi^2\right]$$

$$\mathcal{L}(\phi, \partial_\mu \phi, x^\mu) \qquad \begin{cases} x^\mu \rightarrow x'^\mu = x^\mu + \delta x^\mu \\ \phi(x) \rightarrow \phi'(x) = \phi(x) + \delta\phi(x) \end{cases}$$



$$\phi'(x') = \phi(x) + \Delta\phi(x)$$
$$= \phi(x) + \phi'(x) - \phi(x') + \phi(x') - \phi(x)$$
$$= \phi(x) + \delta\phi + (\partial_\mu \phi)\delta x^\mu$$

$$\delta S = \int \mathcal{L}(\phi', \partial_\mu\phi', x'^\mu) d^4x' - \int \mathcal{L}(\phi, \partial_\mu\phi, x^\mu) d^4x$$

$$d^4x' = J(x', x) d^4x$$

$$J(x';x) \Rightarrow \frac{\partial x'^\mu}{\partial x^\mu} = \delta^\mu{}_\lambda + \partial_\lambda \delta x^\mu$$

$$\det(J) = 1 + \partial_\mu x^\mu$$

$$\delta S = \int (\delta\mathcal{L} + \mathcal{L}\partial_\mu \delta x^\mu) d^4x$$

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \underbrace{\delta(\partial_\mu\phi)}_{\partial_\mu(\delta\phi)} + \frac{\partial\mathcal{L}}{\partial x^\mu}\delta x^\mu$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\delta\phi) = \underbrace{\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right]}_{d^4x \to \text{Stokes}} - \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right]\delta\phi$$

$$\delta S = \int_R \left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right]\right)\delta\phi\, d^4x - \int_{\partial R}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi + \mathcal{L}\delta x^\mu\right)\cdot ds = 0$$

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right] = 0 \quad \Leftrightarrow \quad (m^2 + \Box)\phi = 0$$

$$\int_{\partial R}\left\{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\underbrace{[\delta\phi + (\partial_\nu\phi)\delta x^\nu]}_{\Delta\phi} - \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \delta^\mu{}_\nu\mathcal{L}\right]\delta x^\nu\right\} dS_\mu$$

$$\Theta^\mu{}_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \delta^\mu{}_\nu\mathcal{L}$$

☐ Gauge transformation   Noether's theorem.

$$\begin{cases} \Delta x^\mu = X^\mu{}_\nu \delta\omega^\nu \\ \Delta\phi = \Phi_\mu \delta\omega^\mu \end{cases} \Rightarrow \int_{\partial R}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Phi_\nu - \Theta^\mu{}_\kappa X^\kappa{}_\nu\right)\delta\omega^\nu d\sigma_\mu$$

$$\partial_\mu J^\mu{}_\nu = 0$$

$$Q = \int_\sigma J^\mu{}_\nu d\sigma_\mu \quad \text{charge} \quad \frac{dQ}{dt} = 0$$

$$\begin{cases} \Delta x^\mu = \varepsilon^\mu \\ \Delta\phi = 0 \end{cases} \quad \begin{matrix} X^\mu{}_\nu = \delta^\mu{}_\nu \\ \Phi_\mu = 0 \end{matrix}$$

$$J^\mu{}_\nu = -\Theta^\mu{}_\nu$$

$$P_\nu = \int \Theta^0{}_\nu d^3x \qquad \frac{d}{dt}P_\nu = 0$$

momentum

$$E = \int \Theta^0{}_0 d^3x = \int \left(\frac{\partial\mathcal{L}}{\partial\dot\phi}\dot\phi - \mathcal{L}\right) d^3x$$

$$\frac{dE}{dt} = 0$$

Rotation:

$$\begin{cases} \delta x^i = \varepsilon^{ij} x^j , \quad \varepsilon^{ij} = -\varepsilon^{ji} \\ \Delta\phi = 0 \qquad \delta x^\mu = X^\mu_{\rho\sigma}\varepsilon^{\rho\sigma} \end{cases}$$

$$X^\mu{}_{\rho\sigma} = \frac{1}{2}(\delta^\mu_\rho x_\sigma - \delta^\mu_\sigma x_\rho)$$

$$J^{\mu\rho\sigma} = -T^\mu{}_\kappa X^{\kappa\rho\sigma}$$

$$= \frac{1}{2}(T^{\mu\rho}x^\sigma - T^{\mu\sigma}x^\rho)$$

$$\partial_\mu J^{\mu\rho\sigma} = 0$$

$$M^{\mu\nu} = \int (T^{0\mu}x^\nu - T^{0\nu}x^\mu)d^3x \qquad \text{Angular - momentum}$$

$$\frac{d}{dt}M^{\mu\nu} = 0$$

$$M^{\rho\mu\nu} = T^{\rho\mu}x^\nu - T^{\rho\nu}x^\mu$$

$$\partial_\rho T^{\rho\alpha} = 0 \Rightarrow T^{\mu\nu} = T^{\nu\mu}$$

$$\theta^{\mu\nu} = (\partial^\mu\phi)(\partial^\nu\phi) - g^{\mu\nu}\mathcal{L}$$

Canonical Energy-momentum tensor

$$T^{\mu\nu} = \theta^{\mu\nu} + \partial_\lambda f^{\lambda\mu\nu}$$

$$\partial_\lambda f^{\lambda\nu\mu}. \qquad f^{\lambda\mu\nu} = -f^{\lambda\nu\mu}$$

$$\partial_\nu\partial_\lambda f^{\lambda\mu\nu} = 0$$

3.3. Complex scalar fields.

ϕ_1 , ϕ_2

$$\begin{cases} \phi = (\phi_1 + i\phi_2)/\sqrt{2} \\ \phi^* = (\phi_1 - i\phi_2)/\sqrt{2} \end{cases} \qquad \mathcal{L} = (\partial_\mu\phi)(\partial^\mu\phi^*) - m^2\phi\phi^* \qquad \begin{cases} (m^2 + \Box)\phi = 0 \\ (m^2 + \Box)\phi^* = 0 \end{cases}$$

$$U(1): \qquad \phi \to e^{-i\Lambda}\phi \qquad \phi^* \to e^{i\Lambda}\phi^*$$

$$J^\mu = i(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*)$$

$$\partial_\mu J^\mu = 0$$

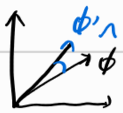
$$Q = \int J^0 dV = i\int(\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t})dV$$

New way to approach.

$$\phi = i\phi_1 + j\phi_2$$

$$\mathcal{L} = (\partial_\mu\phi)\cdot(\partial^\mu\phi) - m^2\phi\cdot\phi$$

$$\begin{cases} \phi_1' + i\phi_2' = e^{-i\Lambda}(\phi_1 + i\phi_2) \\ \phi_1' - i\phi_2' = e^{i\Lambda}(\phi_1 - i\phi_2) \end{cases}$$



$$\text{Local} \to \Lambda \ll 1$$

$$\phi \to \phi - i\Lambda\phi$$

$$\begin{cases} \delta\phi = -i\Lambda\phi \\ \partial_\mu\phi \Rightarrow \partial_\mu\phi - i(\partial_\mu\Lambda)\phi - i\Lambda(\partial_\mu\phi) \end{cases}$$

$$\delta\mathcal{L} = \delta[(\partial_\mu\phi)(\partial^\mu\phi^*)] - m^2\delta(\phi^*\phi)$$
$$= \underline{\underline{(\partial_\mu\Lambda)J^\mu}}\,.$$

New term A_μ

$$\begin{cases} A_\mu \to A_\mu + \frac{1}{e}\partial_\mu\Lambda \\ \phi(x) \to \phi(x)\,e^{ie\Lambda} \end{cases}$$

$$\mathcal{L}_1 = -eJ^\mu A_\mu = -ie(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*)A_\mu$$

$$\delta\mathcal{L}_1 = -e(\delta J^\mu)A_\mu - \underline{J^\mu\partial_\mu A_\mu}$$

$$\mathcal{L}_2 = e^2A_\mu A^\mu\phi^*\phi$$

$$\delta\mathcal{L}_2 = 2eA_\mu(\partial^\mu\Lambda)\phi^*\phi$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\mathcal{L}_3 = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

$$\mathcal{L}_{tot} = \mathcal{L} + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3$$
$$= (\partial_\mu\phi + ieA_\mu)(\partial^\mu\phi^* - ieA^\mu\phi^*) - m^2\phi^*\phi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

J^μ invariant.

$D_\mu\phi$ covariant derivative

$$D_\mu\phi = (\partial_\mu + ieA_\mu)\phi$$

$$\delta(D_\mu\phi) = -i\Lambda(D_\mu\phi)$$

$$\partial_\mu \to (\partial_0, \nabla)$$
$$\nabla \to \nabla - ieA$$

$$p = -i\hbar\nabla \to p \to P = p - eA$$

For a particle with e: $\frac{\partial L}{\partial \dot{q}}$

$$P = p + eA = \gamma m v + eA$$

$$H = \frac{1}{2m}(P - eA)^2 + e\phi$$

$$\Rightarrow D_\mu\phi = (\partial_\mu + ieA_\mu)\phi$$

$$D_\mu\phi^* = (\partial_\mu - ieA_\mu)\phi^*$$

$e \to -e$

Euler-Lagrange equation: $\frac{\partial\mathcal{L}}{\partial A_\mu} - \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right] = 0$

$$\partial_\mu F^{\mu\nu} = -ie(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*) + 2e^2 A^\mu \phi^* \phi$$

$$= -ie(\phi^* D^\mu \phi - \phi D^\mu \phi^*)$$

$$= -e\, \mathcal{J}^\mu \qquad \partial_\mu \mathcal{J}^\mu = 0$$

Covariant current