YM理论

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1. Lie Algebra & Lie Group.

Basic concepts:

Lie group is a group with infinite elements, and also a differentiable manifold. Any of these elements can be written as:

$$U = \exp[\,i\theta^a T^a\,]\cdot \mathbb{1}.$$

1 is identity.

θ^a are parameters

T^a are generators of this Lie group.

Generators, T^a, form a Lie Algebra by defining multiplication: $[\,,\,]: G\times G \to G$ $\,[T^a, T^b] = i f^{abc} T^c$.

f^{abc} are known as "structure constants". $f^{abc}=0 \Leftrightarrow$ Abelian Lie group, while $f^{abc}\neq 0 \Leftrightarrow$ Non-Abelian Lie group.

"$[\,.\,]$" is called Lie bracket. Sometimes we call it "commutator", implying: $[A, B] = AB - BA$.

Hint: $[A, B] = AB - BA$ is a well-defined group element, while 'AB' or 'BA' is not.

Lie braket seems antisymmetric, so Jacobi identity: $[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$.

Representation:

ideal: an ideal of Lie algebra G is a subalgebra $I\subset G$, satisfying: $\forall\, i\in I,\ g\in G,\ [g, i]\subset I$.

simple: if a Lie algebra has no non-trivial ideals, then it's simple. e.g.: $su(N)$, $so(N)$ are simple.

semi-simple: direct sum of several simple algebras. e.g. the standard model algebra $su(3)\oplus su(2)\oplus u(1)$, is semi-simple.

fundamental repr: the smallest non-trivial repr. its Hermitian conjugate trans under Anti-fundamental repr.

for $SU(N)$, $N\times N$ Hermitian matrices with $\det = 1$.

$$\alpha^a\in\mathbb{R}.\quad \phi_i \to \phi_i + i\alpha^a (T^a_{+})_{ij}\phi_j$$

$$\phi_i^* \to \phi_i^* + i\alpha^a (T^a_{-})_{ij}\phi_j^*$$

$$= \phi_i^* - i\alpha^a (T^a_{+})_{ji}\phi_j^*$$

H.C. = $*\circ T$.

e.g. $SU(2)$. $T^a = \frac{\sigma^a}{2}$ (Pauli Matrices). $f^{abc} = \varepsilon^{abc}$.

$SU(3)$ $T^a = \frac{\lambda^a}{2}$ (Gell-Mann Matrices). f^{abc} ....

What's the normalization relation?

$$f^{acd} f^{bcd} = N\delta^{ab}. \Rightarrow \operatorname{tr}(T^a T^b) = \tfrac{1}{2}\delta^{ab}.$$

adjoint repr.: acts on the vector spaces spanned by generators

since $SU(N)$ has N^2-1 generators. adjoint repr. gives $(N^2-1)\times(N^2-1)$ matrices.

for $su(2)$, we still have $f^{abc} = \varepsilon^{abc}$. $[T^a_{adj}, T^b_{adj}] = i f^{abc} T^c_{adj}$.

Casimir Operator: suppose $R = $ adj or $R = $ fund. T^a_R is a repr of group elements. Casimir operator C_R (under a certain repr.) is an operator that commutes with all group elements. Here we have Schur's Lemma:

Schur's Lemma: a group element which commutes with all other elements in any irreducible repr. must be proportional to identity $\mathbb{1}$.

Thus, we may guess a Casimir operator $C_2(R)$ depending on explicit repr. (called quadratic Casimir).

$$T^a_R T^a_R = C_2(R)\mathbb{1}.$$

check:
$$\begin{aligned}[T^a_R T^a_R, T^b_R] &= T^a_R[T^a_R, T^b_R] + [T^a_R, T^b_R]T^a_R \\ &= T^a_R\cdot i f^{abc} T^c_R + i f^{abc} T^c_R T^a_R \\ &= i f^{abc}\{T^a_R, T^c_R\}. \qquad \text{(a.c) sym in } \{T^a_R, T^c_R\} \text{ while anti-sym in } f^{abc}. \\ &= 0\end{aligned}$$

Now we generally consider the identity: $\operatorname{tr}(T^a_R T^b_R) = T(R)\delta^{ab}$, $T(R)$ is called "index" of the repr.

$$\begin{cases} T(\text{adj}) = N \\ T(\text{fund}) = \frac{1}{2}. \end{cases}$$

let $a=b$, sum over a:

$$\underbrace{C_2(R)}_{\text{diag element}}\cdot \underbrace{d(R)}_{\text{dimension}} = T(R)\,\underbrace{d(G)}_{\text{also = number of generators}}$$

for $R = $ adj. $C_2(\text{adj}) = \dfrac{T(\text{adj})\,d(G)}{d(\text{adj})} = \dfrac{N(N^2-1)}{N^2-1} = N$, $\triangleq C_A \to \operatorname{tr}(T^a T^b) = T_F\delta^{ab}$

$R = $ fund. $C_2(\text{fund}) = \dfrac{T(\text{fund})\,d(G)}{d(\text{fund})} = \dfrac{\frac{1}{2}(N^2-1)}{N} = \dfrac{N^2-1}{2N}$. $\triangleq C_F \to f^{acd}f^{bcd} = C_A\delta^{ab}$

usually, we use fundamental repr. for calculation.

and: $(T^a T^a)_{ij} = C_F\delta_{ij}$.

This 3 equations give the most important attributes of $SU(N)$ and its repr.

Also, Fierz Identity: $T^a_{ij} T^a_{kl} = \frac{1}{2}\left(\delta_{ik}\delta_{jl} - \frac{1}{N}\delta_{ij}\delta_{kl}\right)$.

2. Gauge invariance.

In gauge theory, a local gauge trans allows scalar field $\phi(x) \to e^{i\alpha(x)}\phi(x)$, with theory itself invariant.

thus: $\phi(y) - \phi(x) \to e^{i\alpha(y)}\phi(y) - e^{i\alpha(x)}\phi(x)$. However, when we try to define derivatives, we got:

$$\partial_\mu\phi(x) \longrightarrow \partial'_\mu\phi(x) = \lim_{y\to x} \frac{e^{i\alpha(y)}\phi(y) - e^{i\alpha(x)}\phi(x)}{y-x} = ? \quad \text{for } \alpha(y) \text{ is arbitrary function.}$$

Here we define a bi-local field: $W(x,y)$, and let it trans by: $\begin{cases} \phi(x) \to e^{i\alpha(x)}\phi(x) \\ W(x,y) \to e^{i\alpha(x)} W(x,y) e^{-i\alpha(y)} \end{cases}$.

Then we have $W(x,y)\phi(y) - \phi(x) \to e^{i\alpha(x)} W(x,y) e^{-i\alpha(y)} e^{i\alpha(y)}\phi(y) - e^{i\alpha(x)}\phi(x) = e^{i\alpha(x)}[W(x,y)\phi(y) - \phi(x)]$.

$W(x,y)\phi(y) - \phi(x)$ act as a scalar field under gauge trans.
and also, $W(x,y)$ seems like a pull-back.

We can define derivatives: $D_\mu\phi(x) = \lim_{\delta x\to 0} \frac{W(x,x+\delta x)\phi(x+\delta x) - \phi(x)}{\delta x^\mu}$ ($D_\mu\phi(x) \to e^{i\alpha(x)} D_\mu\phi(x)$).

Set $W(x,x) = 1$, and expand it for infinitsimal δx: $W(x, x+\delta x) = 1 - ie\delta x^\mu A_\mu + O(\delta x^2)$. e is an arbitrary constant.

$$\begin{aligned} W(x, x+\delta x) \to e^{i\alpha(x)} W(x, x+\delta x) e^{-i\alpha(x+\delta x)} &= (1+i\alpha(x))(1 - ie\delta x^\mu A_\mu)(1 - i\alpha(x+\delta x)) \\ &= (1+i\alpha(x))(1 - ie\delta x^\mu A_\mu)(1 - i\alpha(x) - i\delta x^\mu \partial_\mu\alpha(x)) \\ &= 1 - ie\delta x^\mu A_\mu - i\delta x^\mu \partial_\mu\alpha(x) \end{aligned}$$

$\Rightarrow A_\mu \to A_\mu + \frac{1}{e}\partial_\mu\alpha$. that's how $A(x)$ trans under this gauge trans. $W(x,y)$ is called 'Wilson line'.

$D_\mu\phi = \partial_\mu\phi - ieA_\mu\phi(x)$. A_μ seems like a connection like Christoffel symbol (with only 1 index).

Closed form of $W(x,y)$ can be written: $W_P(x,y) = \exp(ie\int_y^x A_\mu(z)dz^\mu)$. the integral along the path: $P: x\to y$.

parameterize the path by $z^\mu(\lambda)$. $\lambda=0$ for y^μ and $\lambda=1$ for x^μ. $W_P(x,y) = \exp(ie\int_0^1 A_\mu(z(\lambda))\frac{dz^\mu}{d\lambda}\cdot d\lambda)$

A special case is that $P: x\to x$ as a contour. then $W_P(x,y) \to W_P^{loop} = \exp(ie\oint A_\mu(x)dx^\mu)$. named 'Wilson loop'.

By Stoke's Theorem. we know $\int_{\partial D} w = \int_D dw$. in Abelian gauge theory. $[A_\mu, A_\nu] = 0$. thus $F = \frac{1}{2}F_{\mu\nu}dx^\mu\wedge dx^\nu$ (curvature 2-form) is just the differential of $A = A_\mu dx^\mu$ (connection 1-form). So: $W_P^{loop} = \exp(i\frac{e}{2}\int_\Sigma F_{\mu\nu}d\sigma^{\mu\nu})$. surface Σ is what P bounds.

D_μ is now the covariant derivative. so $[D_\mu, D_\nu]$ is still a derivative (but not 2-nd order derivative).

$$\begin{aligned} [D_\mu, D_\nu]\phi(x) &= (\underbrace{[\partial_\mu, \partial_\nu]}_{=0}\phi(x) - e^2\underbrace{[A_\mu, A_\nu]}_{=0}\phi(x) - ie[\partial_\mu, A_\nu]\phi(x) + ie[\partial_\nu, A_\mu]\phi(x)). \\ &= ie[\partial_\nu A_\mu - A_\mu\partial_\nu + A_\nu\partial_\mu - \partial_\mu A_\nu] \\ &= -2ie(\partial_\mu A_\nu - \partial_\nu A_\mu)\phi(x). \\ &= -ie F_{\mu\nu}\phi(x). \end{aligned}$$

we redefined $F = F_{\mu\nu}dx^\mu\wedge dx^\nu = \frac{1}{2}F^{\mu\nu}dx^\mu\wedge dx^\nu$.
2 is just a constant.

This implies $F_{\mu\nu} = \frac{i}{e}[D_\mu, D_\nu]$ is just the field strength 2-form.

3. Non-Abelian Gauge.

For example. we consider N Dirac-Fermions lagrangian: $\mathcal{L} = \sum_{j=1}^N \bar\psi_j(i\not\partial - m)\psi_j$

under a global SU(N) symmetry, $\psi_j \to (e^{i\alpha^a T^a})_{ij}\psi_j$. T^a are SU(N) generators in fund. repr.

Wilson line must be written as: $W_P(x,y) = P\{\exp(ig\int A_\mu^a T^a dz)\}$. $P\{\cdots\}$ is the path-ordering operator to encounter $[A_\mu^a, A_\nu^a] \neq 0$.

Explicitly: $W_P(x,y) = 1 + ig\cdot\int_0^1 A_\mu^a(z(\lambda))T^a\frac{dz^\mu}{d\lambda}d\lambda - \frac{1}{2}g^2\cdot\int_0^1 d\lambda\int_0^1 d\tau\, A_\mu^a(z(\lambda))A_\nu^b(z(\tau))[T^aT^b\theta(\lambda-\tau) + T^bT^a\theta(\tau-\lambda)]\frac{dz^\mu}{d\lambda}\frac{dz^\mu}{d\tau} + \cdots$

Still we have: $W_P(x,y) \to e^{i\alpha^a(x)T^a} W(x,y) e^{-i\alpha^a(y)T^a}$ ← $T^{a\dagger}$ actually, while for SU(N), generators are Hermitian.

in non-Abelian gauge theory, we take $A_\mu = A_\mu^a T^a$. the 1-form: $A = A_\mu dx^\mu$. $W_P(x,y) = P\{\exp(ig\int_y^x A_\mu(z)dz^\mu)\}$.

infinitesimal expansion: $W(x, x+\delta x) = 1 - igA_\mu\delta x^\mu$.

Now, consider $U(x) = e^{i\alpha^a(x)T^a} \in SU(N)$. this implies a local SU(N) gauge trans: $\vec\psi(x) \to U(x)\vec\psi(x)$.

$W(x,y) \to U(x)W(x,y)U^\dagger(y)$.

for covariate derivative D_μ: $D_\mu\vec\psi(x) \to U(x)D_\mu\vec\psi(x) = D'_\mu U(x)\vec\psi(x)$.

this equation implies the equivalence of initiative and passive viewpoint of a certain transformation. $UD_\mu\vec\psi$ is the former while $D'_\mu(U\vec\psi)$ is the latter.

$\Rightarrow U(\partial_\mu - igA_\mu)\vec\psi = (\partial_\mu - igA'_\mu)(U\vec\psi)$

$\Rightarrow -igUA_\mu = \partial_\mu U - igA'_\mu U$. thus we know $A'_\mu = UA_\mu U^\dagger - \frac{i}{g}(\partial_\mu U)U^\dagger$.

$\Leftrightarrow A_\mu^{a\prime} = A_\mu^a + \frac{1}{g}\partial_\mu\alpha^a - f^{abc}\alpha^b A_\mu^c$. ($O(\alpha^2)$ dropped).

for Lie-bracket: $[D_\mu, D_\nu]\vec\psi(x) = (-ig(\partial_\mu A_\nu - \partial_\nu A_\mu) - g^2[A_\mu, A_\nu])\vec\psi(x)$.

then we define field strength: $F_{\mu\nu} \equiv \frac{i}{g}[D_\mu, D_\nu] = (\partial_\mu A_\nu - \partial_\nu A_\mu) - ig[A_\mu, A_\nu]$ > for general non-Abelian case.

$\Leftrightarrow F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c$. and $F_{\mu\nu}^a$ trans by: $F_{\mu\nu}^{a\prime} = F_{\mu\nu}^a - f^{abc}\alpha^b F_{\mu\nu}^c$.

all these $-f^{abc}\alpha^b A^c$ term comes from extracting A^a from A^aT^a term.
Remember $[D_\mu, D_\nu] \to U[D_\mu, D_\nu]$. the same as $A_\mu \to UA_\mu$.

4. Lagrangian and physics.

$\underline{\quad\quad}$. from extracting A^a from $A^a T^a$ term.
Remember $[D_\mu, D_\nu] \to U[D_\mu, D_\nu]$, the same as $A_\mu \to UA_\mu$.

4. Lagrangian and physics.

First we take locally $SU(N)$ invariant lagrangian: $\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \sum_{i,j=1}^{N}\bar\psi_i(\delta_{ij}i\not\partial + g\not A^a T^a_{ij} - m\delta_{ij})\psi_j$.

explicitly: $\mathcal{L} = -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu)^2 + \sum_{i,j=1}^{N}\bar\psi_i(i\delta_{ij}\gamma^\mu\partial_\mu + g\gamma^\mu A^a_\mu T^a_{ij} - m\delta_{ij})\psi_j$.

$$\Rightarrow \begin{cases} \partial_\mu F^{a\mu\nu} + gf^{abc}A^b_\mu F^{c\mu\nu} = -g\bar\psi_i\gamma^\nu T^a_{ij}\psi_j & \text{EoM for gauge field } A^a_\mu. \\ (i\not\partial - m)\psi_i = -g\not A^a T^a_{ij}\psi_j. & \text{EoM for spinor.} \end{cases}$$

Why don't we use $F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu$ to construct F? Because D_μ covariant is for $\phi \to e^{i\alpha(x)}\phi$, but not for A_μ.

for global gauge trans: $\begin{cases} \psi_i \to \psi_i + i\alpha^a T^a_{ij}\psi_j \Leftrightarrow \bar\psi_i \to \bar\psi_i - i\alpha^a\bar\psi_j T^a_{ji} \\ A^a_\mu \to A^a_\mu - f^{abc}\alpha^b A^c_\mu. \end{cases}$ In non-Abelian case, we get N^2-1 currents, one for each α^a.

Noether's Theorem claims $J^\mu = \sum_{\text{fields}} \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\frac{\delta\phi}{\delta\alpha}$ is a conserved current, now we get: $J^{a\mu} = \sum_{b=1}^{N^2-1}\frac{\partial\mathcal{L}}{\partial(\partial_\mu A^b_\nu)}\frac{\delta A^b_\nu}{\delta\alpha^a} + \sum_{i=1}^{N}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\frac{\delta\psi_i}{\delta\alpha^a}$

$$= f^{abc}A^b_\nu F^{c\mu\nu} - \bar\psi_i\gamma^\mu T^a_{ij}\psi_j$$

Separate $J^{a\mu}$ to $k^{a\mu} = f^{abc}A^b_\nu F^{c\mu\nu}$, $j^{a\mu} = -\bar\psi_i\gamma^\mu T^a_{ij}\psi_j$. Apparently, $\partial_\mu J^{a\mu} = 0$, but $\partial_\mu k^{a\mu} \neq 0$, $\partial_\mu j^{a\mu} \neq 0$.
We can formally define conserved current/charge, but they are not gauge invariant or gauge covariant.

Actually, we have Weinberg-Witten Theorem:

Weinberg-Witten Theorem:

1. global non-Abelian symmetry + charged massless spin-1 $\not\longrightarrow$ gauge invariant conserved charge.
2. conserved & Lorentz-covariant energy-momentum tensor $\not\longleftrightarrow$ massless spin-2

But something important: W-W Theorem assumes fixed space-time dimension, so it doesn't forbid string theory and AdS/CFT to build gravity in separated dimension of space-time.

4. Feynman Rules for YM Theory:

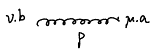
Now we completely write down the $SU(N)$ invariant Lagrangian with non-Abelian gauge fields:

$$\mathcal{L} = \underbrace{-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}}_{\text{non-Abelian fields}} \underbrace{- \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2}_{R\text{-}\xi\text{ gauge term}} + \underbrace{(\partial^\mu\bar c^a)(\delta^{ac}\partial_\mu + f^{abc}A^b_\mu)c^c}_{\text{Faddeev-Popov ghosts}} + \bar\psi_i(i\not\partial - m)\psi_i + \underbrace{\bar\psi_i g\not A^a T^a_{ij}\psi_j}_{\text{Fermion-gauge interaction}} + [(\delta_{ki}\partial^\mu \underbrace{- igA^{a\mu}T^a_{ki})\phi_i]^*[(\delta_{kj}\partial_\mu - igA^a_\mu T^a_{kj}}_{\text{including boson-gauge interaction}})\phi_j] - M^2\phi_i^*\phi_i.$$

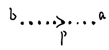
$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc}A^b_\mu A^c_\nu.$$

Kinetic terms: $\mathcal{L}_{kin} = -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 + \bar\psi_i(i\not\partial - m)\psi_i - \phi_i^*(\Box + M^2)\phi_i - \bar c^a\Box c^a$.

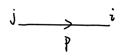
$\Rightarrow$ propagator:



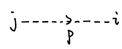
$= \frac{-ig^{\mu\nu}}{p^2 + i\varepsilon}\delta^{ab}$ $(\xi = 1)$. gauge boson propagator (e.g. gluons).



$= \frac{i\delta^{ab}}{p^2 + i\varepsilon}$. ghost propagators



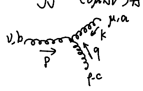
$= \frac{i\delta^{ij}}{\not p - m + i\varepsilon}$ colored fermions propagators.



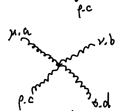
$= \frac{i\delta^{ij}}{p^2 - M^2 + i\varepsilon}$ colored bosons propagators

Interaction terms: $\mathcal{L}_{int} = -gf^{abc}(\partial_\mu A^a_\nu)A^{b\mu}A^{c\nu} - \frac{1}{4}g^2(f^{eab}A^{a\mu}A^{b\nu})(f^{ecd}A^c_\mu A^d_\nu) + gf^{abc}(\partial^\mu\bar c^a)A^b_\mu c^c + gA^a_\mu\bar\psi_i\gamma^\mu T^a_{ij}\psi_j + igA^{a\mu}T^a_{ij}(\phi_i^*\partial_\mu\phi_j - \phi_j\partial_\mu\phi_i^*) + g^2\phi_i^*A^a_\mu T^a_{ik}T^b_{kj}A^b_\mu\phi_j$.

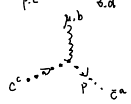
$\Rightarrow$ Vertex:



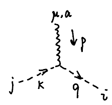
$= gf^{abc}[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\nu + g^{\rho\mu}(q-k)^\nu]$. $p+k+q=0$.



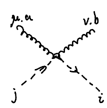
$= -ig^2[f^{abe}f^{cde}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})]$



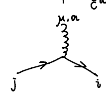
$= -gf^{abc}p^\mu$.



$= ig(k^\mu + q^\mu)T^a_{ij}$.



$= ig^2 T^a_{ik}T^b_{kj}g^{\mu\nu}$.



$= ig\gamma^\mu T^a_{ij}$.