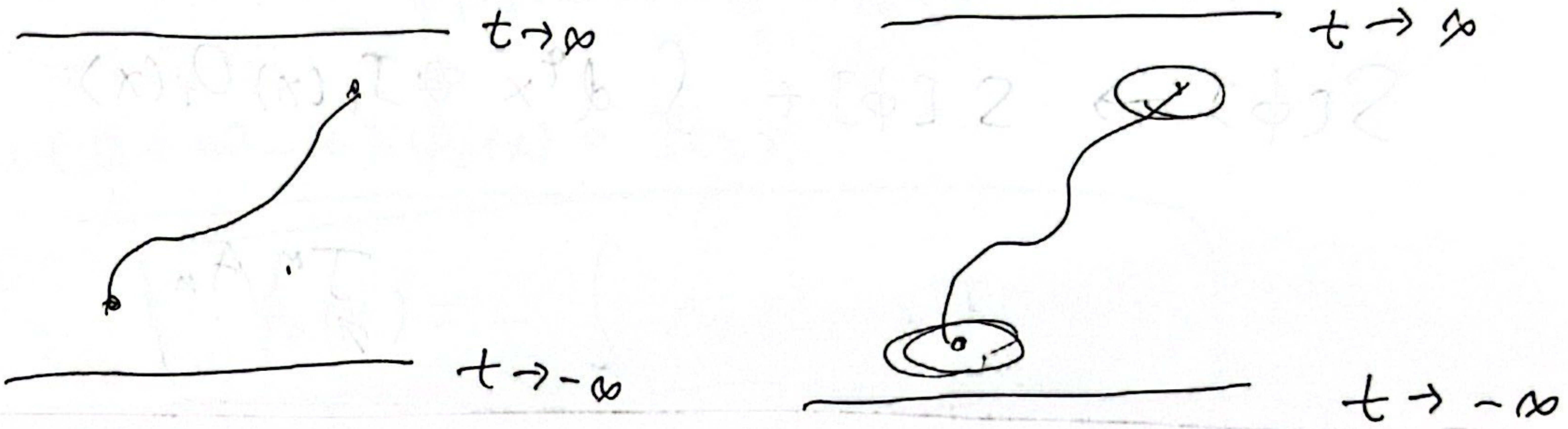
1 Generating functionals

Review: $\mathcal{H} := \mathrm{span}\{|\phi, t\rangle\}$, $1 = \int \mathcal{D}\phi\, |\phi, t\rangle\langle\phi, t|$

Principle:

$$\langle q_f, t_f | q_i, t_i \rangle = N \int \mathcal{D}q\, \exp\left[\frac{i}{\hbar}\int_{t_i}^{t_f} L\, dt\right].$$

$$\langle \phi_f, t_f | \phi_i, t_i \rangle = N \int \mathcal{D}\phi\, \exp\left[\frac{i}{\hbar}\int \mathcal{L}\, d^4x\right]$$



$$L \to L + \hbar J(t)\, q(t)$$

$$\langle 0, \infty | 0, -\infty \rangle^J \propto \int \mathcal{D}q\, \exp\left[\frac{i}{\hbar}\int_{-\infty}^{+\infty} dt\, \left(L + \hbar J q + \frac{1}{2} i\epsilon q^2\right)\right]$$

$$e^{-iE_n(T'-T)}$$

Feyman's $i\epsilon$ prescription

$$\int \mathcal{D}\phi\, \exp\left[\frac{i}{\hbar}\int_{-\infty}^{+\infty} d^4x\, \left(\mathcal{L} + \hbar J\phi + \frac{1}{2} i\epsilon\phi^2\right)\right]$$

$$T \to T(1 + i\epsilon)$$

$$= Z[J]$$

$$\left.\frac{\delta^n Z[J]}{\delta J(t_1)\cdots\delta J_n(t_n)}\right|_{J=0} \propto i^n \langle 0, \infty | T[q(t_1)\cdots q(t_2)] | 0, -\infty \rangle$$

Partition function

$$Z = \int_C D\phi \ \exp\left(-\frac{1}{\hbar} S[\phi]\right) \qquad Z = \sum_n e^{-\beta E_n}$$

$$\langle \phi_f | \phi_i \rangle = \frac{1}{Z} \int_{C(\phi_i, \phi_f)} D\phi \ e^{i S[\phi]/\hbar}$$

$$\langle O_1(x_1) O_2(x_2) \cdots O_n(x_n) \rangle = \frac{(-\hbar)^n}{Z} \frac{\delta Z[J]}{\delta J_1(x_1)\, \delta J_2(x_2) \cdots \delta J_n(x_n)}$$

$$S[\phi] \rightarrow S[\phi] + \int d^4x \ J_i(x) O_i(x)$$

$$J^\mu A_\mu$$

Scalar field

$$Z[J] = \int D\phi \ \exp\left\{ i \int d^4x \left[\mathcal{L}(\phi) + J(x)\phi(x) + \frac{1}{2} i\epsilon \phi^2 \right] \right\}$$

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi - m^2 \phi) \qquad \partial_\mu \phi \partial^\mu \phi = \partial_\mu(\phi \partial^\mu \phi) - \phi \Box \phi$$

$$= \int D\phi \ \exp\left(i \int d^4x \left\{ \frac{1}{2} [\partial_\mu \phi \partial^\mu \phi - (m^2 - i\epsilon)\phi^2] + \phi J \right\} \right)$$

$$= \int D\phi \ \exp\left(-i \int d^4x \left\{ \frac{1}{2} \phi(\Box + m^2 - i\epsilon)\phi - \phi J \right\} \right)$$

$$\int e^{-ax^2+bx}\,dx$$

$$\int \exp\left\{-\tfrac{1}{2}x^T A x\right\} dx = (\det A)^{-1/2}$$

A is a real, positive-definite matrix

$$\int \exp\left\{-\tfrac{1}{2}x^T A x + b\cdot x + c\right\} dx \qquad d^n x$$

$$= \exp\left[\tfrac{1}{2}b^T A^{-1} b - c\right](\det A)^{-1/2}$$

$$\int D\phi\, \exp\left[-\tfrac{1}{2}\phi\, A(x)\, \phi\right] = (\det A)^{-1/2}$$

$$Z_0[J] = \exp\left[\tfrac{i}{2}\int J(x)(\Box + m^2 - i\epsilon)^{-1} J(y)\,dx\,dy\right]$$

$$\left[\det i(\Box + m^2 - i\epsilon)\right]^{-1/2}$$

$$\boxed{(\Box + m^2 - i\epsilon)^{-1} = -\Delta_F(x-y)}$$

$$(\Box + m^2 - i\epsilon)\Delta_F(x) = -\delta^4(x)$$

$$\Delta_F(x) = -\frac{\delta^4(x)}{\Box + m^2 - i\epsilon}$$

$$\Delta_F(x-y) = -\frac{\delta^4(x-y)}{\Box + m^2 - i\epsilon}$$

$$Z_0[J] = \exp\left[-\tfrac{i}{2}\int J(x)\,\Delta_F(x-y)\,J(y)\,d^4x\,d^4y\right],$$

$$= 1 - \tfrac{i}{2}\int J(x)\,\Delta_F(x-y)\,J(y)\,d^4x\,d^4y + \cdots$$

$\int D\phi \qquad \phi(x) \to \phi(x) + \phi_0(x)$

$\exp \int \left[\frac{1}{2}\phi(\Box+m^2-i\epsilon)\phi - \phi J\right] d^4x$

$= \exp\int\left[\frac{1}{2}\phi(\Box+m^2-i\epsilon)\phi + \phi(\Box+m^2-i\epsilon)\phi_0 + \frac{1}{2}\phi_0(\Box+m^2-i\epsilon)\phi_0 - \cancel{\phi J} - \phi_0 J\right] d^4x$

$\phi(\Box+m^2-i\epsilon)\phi_0 = \phi_0(\Box+m^2-i\epsilon)\phi$

choose $(\Box+m^2-i\epsilon)\phi_0(x) = J(x)$

$$\boxed{\phi_0(x) = -\int \Delta_F(x-y)J(y)\,dy}$$

$(\Box+m^2-i\epsilon)\Delta_F(x) = -\delta^4(x)$

$(\Box+m^2-i\epsilon)\int D_F(k)e^{-ikx}dk = \frac{1}{(2\pi)^4}\iint -\delta^4(x)e^{-ikx}dx\, e^{ik'x}dk'$

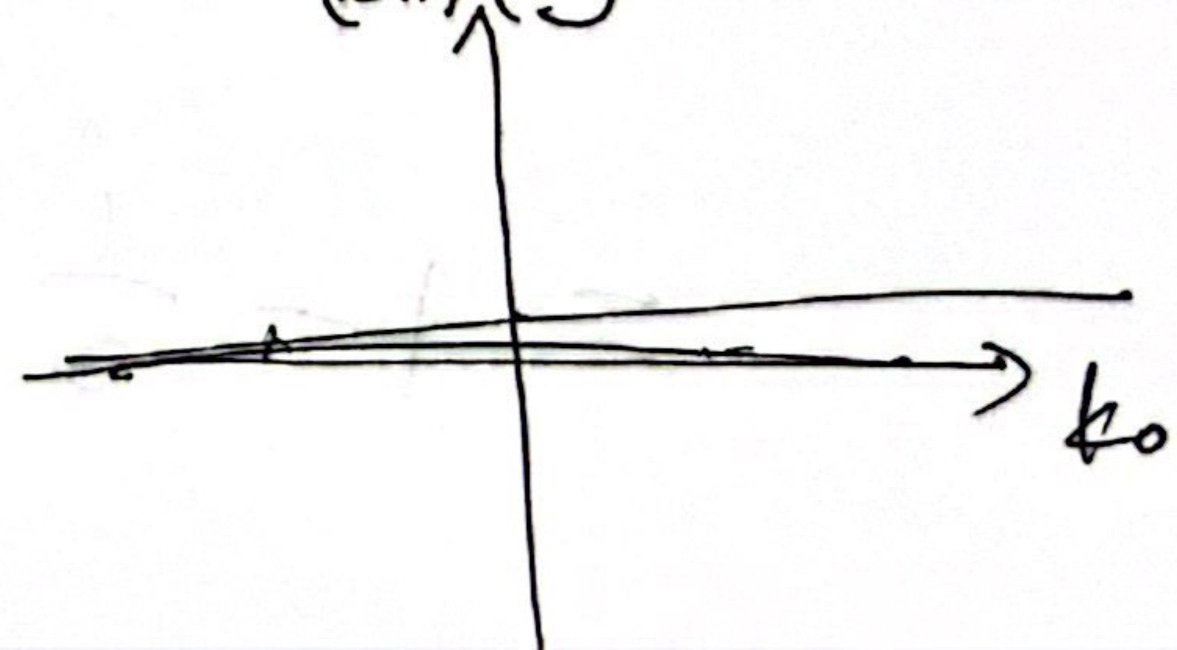
$= \int D_F(k)(k^2+m^2-i\epsilon)e^{-ikx}dk \qquad = \cancel{\frac{1}{(2\pi)^4}\int \delta^4(x)\, e^{i(k'-k)x}\,dx\,dk'}$

$= \cancel{\frac{1}{(2\pi)^4}\delta}$

$\Rightarrow D_F(k) = \frac{1}{k^2-m^2+i\epsilon} \qquad = -\frac{1}{(2\pi)^4}\int e^{-ikx}dk$

$\Delta_F(x) = \frac{1}{(2\pi)^4}\int d^4k \frac{e^{-ikx}}{k^2-m^2+i\epsilon}$

$x^4 = it = ix_0 \qquad k_0$



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$$= \exp\left(\int \left(\tfrac{1}{2}\phi(\Box + m^2 - i\epsilon)\phi - \tfrac{1}{2}\phi_0 J\right) d^4x\right)$$

$$= \exp\left(\tfrac{1}{2}\int \phi(\Box + m^2 - i\epsilon)\phi \, d^4x\right) \exp\left(\tfrac{1}{2}\int J(x)\Delta_F(x-y) J(y)\, dy\, dx\right)$$

$$Z_0[J] = N \exp\left[-\tfrac{i}{2}\int J(x)\,\Delta_F(x-y)\,J(y)\,dx\,dy\right].$$

$$\langle 0|T[\phi(x_1)\phi(x_2)]|0\rangle \propto \frac{1}{i^2}\frac{\delta Z[J]}{\delta J(x_1)\,\delta J(x_2)}\Bigg|_{J=0}$$

$$= \frac{1}{i^2}\frac{\delta}{\delta J(x_1)\,\delta J(x_2)} \exp\left[-\tfrac{i}{2}\int J(x)\,\Delta_F(x-y)\,J(y)\,dx\,dy\right]\Bigg|_{J=0}$$

$$= \frac{1}{i^2}\frac{\delta}{\delta J(x_1)\,\delta J(x_2)}\left(-\tfrac{i}{2}\int J(x)\,\Delta_F(x-y)\,J(y)\,dx\,dy\right)\Bigg|_{J=0}.$$

$$\left(-\tfrac{i}{2}J(x)\,\Delta_F(x-y)\,J(y)\right)$$

$$= -\frac{1}{2i}\Big(\Delta_F(x_2-x_1) + \Delta_F(x_1-x_2)\Big)$$

$$\frac{\delta f(x)}{\delta f(y)} = \delta^4(x-y)$$

$$= -\frac{1}{i}\Delta_F(x_2-x_1)$$

$$= i\,\Delta_F(x_2-x_1)$$