Fermions

费米子场：反对易：$\{\psi(x),\psi(y)\}\big|_{x^0=y^0}=0$

C-number

Grassmann 生成元 ε_i

$$\{\varepsilon_i,\varepsilon_j\}=0$$

其中 i,j 满足 $i,j=1,2,\cdots,n$

$$\boxed{\varepsilon_i^2=0}$$

$$f(\varepsilon_1,\varepsilon_2)=a_0+a_1\varepsilon_1+a_2\varepsilon_2+a_3\varepsilon_1\varepsilon_2$$
$$=a_0+a_1\varepsilon_1+a_2\varepsilon_2-a_3\varepsilon_2\varepsilon_1$$

$$\varepsilon_i^2\varepsilon_j=0$$

定义左微分

$$\frac{\partial f}{\partial\varepsilon_1}=\frac{\partial^L f}{\partial\varepsilon_1}=a_1+a_3\varepsilon_2$$
$$\frac{\partial f}{\partial\varepsilon_2}=\frac{\partial^L f}{\partial\varepsilon_2}=a_2-a_3\varepsilon_1$$

同样也可定义右微分.

$$\frac{\partial^R f}{\partial\varepsilon_1}=a_1-a_3\varepsilon_2$$

注意到

$$\varepsilon_1\frac{\partial f}{\partial\varepsilon_1}=a_1\varepsilon_1+a_3\varepsilon_1\varepsilon_2$$

$$\varepsilon_1 f=a_0\varepsilon_1+a_2\varepsilon_1\varepsilon_2$$

$$\frac{\partial}{\partial\varepsilon_1}(\varepsilon_1 f)=a_0+a_2\varepsilon_2$$

$$\Rightarrow\quad\left(\varepsilon_1\frac{\partial}{\partial\varepsilon_1}+\frac{\partial}{\partial\varepsilon_1}\varepsilon_1\right)f=f$$

或写成

$$\varepsilon_1\frac{\partial}{\partial\varepsilon_1}+\frac{\partial}{\partial\varepsilon_1}\varepsilon_1=1$$

一般地，

$$\{\varepsilon_i, \frac{\partial}{\partial \varepsilon_j}\} = \delta_{ij} \qquad \{\frac{\partial}{\partial \varepsilon_i}, \frac{\partial}{\partial \varepsilon_j}\} = 0$$

$d\varepsilon_i$ 也是 Grassmann 量.

$$\begin{cases} \{\varepsilon_i, d\varepsilon_j\} = 0 \\ \{d\varepsilon_i, d\varepsilon_j\} = 0 \end{cases}$$

$$\int d\varepsilon_1 d\varepsilon_2 f(\varepsilon_1, \varepsilon_2) = \int d\varepsilon_1 \left[\int d\varepsilon_2 f(\varepsilon_1, \varepsilon_2)\right]$$

$$\int d\varepsilon_1 \ , \ \int \varepsilon d\varepsilon_1 = ?$$

$$\begin{aligned} \left(\int d\varepsilon_1\right)^2 &= \int d\varepsilon_1 \int d\varepsilon_2 \\ &= \int d\varepsilon_1 d\varepsilon_2 \\ &= -\int d\varepsilon_2 d\varepsilon_1 \\ &= -\left(\int d\varepsilon_1\right)^2 \end{aligned}$$

因此. $\int d\varepsilon_1 = \int d\varepsilon_2 = 0$

定义 $\int d\varepsilon_1 \varepsilon_1 = 1$

(n维) $\underbrace{\int d\varepsilon_i \varepsilon_i}_{\text{无求和}} = 1 \quad \int d\varepsilon_i = 0$

$f(\varepsilon_1, \varepsilon_2)$，代入 $\int d\varepsilon_1 f$

得

$$\begin{aligned} \int d\varepsilon_1 f &= \int d\varepsilon_1 \left[a_0 + a_1 \varepsilon_1 + a_2 \varepsilon_2 + a_3 \varepsilon_1 \varepsilon_2\right] \\ &= a_0 \int d\varepsilon_1 + a_1 \int d\varepsilon_1 \varepsilon_1 - a_2 \varepsilon_2 \int d\varepsilon_1 + a_3 \varepsilon_2 \int d\varepsilon_1 \varepsilon_1 \\ &= a_1 + a_3 \varepsilon_2 \end{aligned}$$

现在 $2N$ 个可作为相互独立(复) Grassmann 量，因此

$$\int d\eta = \int d\bar\eta = 0 \qquad \int d\eta\,\eta = \int d\bar\eta\,\bar\eta = 1$$

由于 $\eta^2 = \bar\eta^2 = 0$

$$e^{\bar\eta\eta} = 1 - \bar\eta\eta$$

因此

$$\begin{aligned}\int d\bar\eta\, d\eta\, e^{-\bar\eta\eta} &= \underline{\int d\bar\eta\, d\eta} - \int d\bar\eta\, d\eta\, \bar\eta\eta \\ &= 0 + \int d\bar\eta\, d\eta\, \eta\bar\eta \\ &= 1\end{aligned}$$

推广更高维度：2维

$$\eta = \begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} \qquad \bar\eta = \begin{pmatrix}\bar\eta_1\\ \bar\eta_2\end{pmatrix}$$

$$\bar\eta\eta = (\bar\eta^T\eta) = \bar\eta_1\eta_1 + \bar\eta_2\eta_2$$

$$\begin{aligned}\underline{(\bar\eta\eta)^2} &= (\bar\eta_1\eta_1 + \bar\eta_2\eta_2)(\bar\eta_1\eta_1 + \bar\eta_2\eta_2) \\ &= 2\bar\eta_1\eta_1\bar\eta_2\eta_2\end{aligned}$$

$$e^{-\bar\eta\eta} = 1 - (\bar\eta_1\eta_1 + \bar\eta_2\eta_2) + \bar\eta_1\eta_1\bar\eta_2\eta_2$$

代入积分规则，并定义 $d\bar\eta\, d\eta = d\bar\eta_1\, d\eta_1 \cdot d\bar\eta_2 \cdot d\bar\eta_2$

$$\begin{aligned}\int d\bar\eta\, d\eta\, e^{-\bar\eta\eta} &= \int d\bar\eta_1\, d\bar\eta_2\, d\eta_1\, d\eta_2\, \bar\eta_1\eta_1\bar\eta_2\eta_2 \\ &= 1\end{aligned}$$

做变量变换 $\eta = \underline{M}\alpha \qquad \bar\eta = \underline{N}\bar\alpha$

$$\begin{aligned}\eta_1\eta_2 &= (M_{11}\alpha_1 + M_{12}\alpha_2)(M_{21}\alpha_1 + M_{22}\alpha_2) \\ &= (M_{11}M_{22} - M_{12}M_{21})\alpha_1\alpha_2 \\ &= (\det M)\alpha_1\alpha_2\end{aligned}$$

但，若让 $\int d\eta_1\, d\eta_2\, \eta_1\eta_2 = \int d\alpha_1\, d\alpha_2\, \alpha_1\alpha_2$

我们必须 $\underline{d\eta_1\, d\eta_2 = (\det M)^{-1}\, d\alpha_1\, d\alpha_2}$

$$[\det(MN)]^{-1}\int d\bar{\alpha}d\alpha\, e^{-\bar{\alpha}N^T M\alpha} = 1$$

但 $\det MN = \det(M^T N)$ 令 $M^T N = A$

$$\int d\bar{\alpha}\, d\alpha\, e^{-\bar{\alpha}A\alpha} = \det A$$

无限维 Grassmann 代数生成元 $\varepsilon(x)$

$$\begin{cases} \{\varepsilon(x), \varepsilon(y)\} = 0 \\ \dfrac{\partial^{L,R}\varepsilon(x)}{\partial\varepsilon(y)} = \delta(x-y) \\ \int d\varepsilon(x) = 0\ ;\ \int \varepsilon(x)\, d\varepsilon(x) = 1 \end{cases}$$

Dirac 场的 $\mathcal{L}$

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi$$

$$Z_0[\eta,\bar{\eta}] = \frac{1}{N}\int D\bar{\psi}D\psi \exp\{i\int [\bar{\psi}(x)(i\gamma\cdot\partial - m)\psi(x) + \bar{\eta}\psi(x) + \bar{\psi}(x)\eta(x)]\, dx\}$$

$$N = \int D\bar{\psi}D\psi \exp\left[i\int (\bar{\psi}(x)(i\gamma\cdot\partial - m)\psi(x)\, dx\right]$$

定义 $S^{-1} \equiv i\gamma^\mu\partial_\mu - m$

$$\mathcal{L} = \bar{\psi}S^{-1}\psi$$

令 $Q(\psi,\bar{\psi}) = \bar{\psi}S^{-1}\psi + \bar{\eta}\psi + \bar{\psi}\eta$

Q_{min}, $\psi = ?$

$$\psi_m = -S\bar{\eta}\ ,\ \bar{\psi}_m = -\eta S$$

S^{-1}

$$Q_m = -\bar{\eta}S\eta$$

$$Z_0 = \frac{1}{N}\int D\bar\psi D\psi \exp\left\{ i\int [\mathcal{L}_m + (\psi-\psi_m) S^{-1}(\bar\psi - \bar\psi_m)] dx \right\}$$

$$= \frac{1}{N} \exp\left[-i\int \bar\eta(x) S \eta(y)\, dxdy\right] \det(-iS^{-1})$$

$\underbrace{e^{i\mathcal{L}_m}}$ $\psi\bar\psi$

此处 $N = \det(-iS^{-1})$

$$Z_0[\eta,\bar\eta] = \exp\left[-i\int \bar\eta(x) S(x-y)\eta(y)\, dxdy\right]$$

$$S(x) = (i\gamma\cdot\partial + m)\underbrace{\Delta_F(x)}_{\text{Feynman propagator}}$$

$$S^{-1}S = (i\gamma\cdot\partial - m)(i\gamma\cdot\partial + m)\Delta_F(x)$$
$$= [\Box - m^2]\Delta_F(x)$$
$$= \delta^4(x)$$

$$T(x,y) = \frac{\delta^2 Z_0[\eta,\bar\eta]}{\delta\eta(x)\delta\bar\eta(y)}\Big|_{\eta=\bar\eta=0}$$

$$= -\frac{\delta}{\delta\eta(x)}\cdot\frac{\delta}{\delta\eta(y)} \exp\left[-i\int \bar\eta(x) S(x-y)\eta(y)\, dxdy\right]\Big|_{\eta=\bar\eta=0}$$

$$= iS(x-y)$$

标量场 - 玻色场

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 = -\frac{1}{2}\phi(\Box + m^2)\phi$$

两点 (2-point) 函数

$$T(x,y) = i\Delta_F(x-y)$$

此处 Δ_F 满足：

$(\Box + m^2) \Delta_F(x-y) = -\delta^4(x-y)$

$\mathcal{L}_0 = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi = \bar{\psi}S^{-1}\psi$

两点函数：$\tau(x,y) = iS(x-y)$

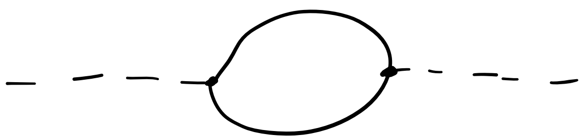
假设了 $\begin{cases} [\xi_i, d\xi_j] = 0 \\ [d\xi_i, d\xi_j] = 0 \end{cases}$

$$\frac{\delta^2}{\delta\eta(x)\delta\eta(y)} = -\frac{\delta^2}{\delta\eta(y)\delta\eta(x)}$$

其中 η 费米场源

左微分

$$\frac{\delta}{\delta\eta(x_1)}[\eta(x)\eta(y)] = \delta^4(x_1-x)\eta(y) - \delta^4(x_1-y)\eta(x)$$



$$\underset{\sim\sim\sim}{Z[\eta,\bar{\eta}]} = \exp\left[i\int \mathcal{L}_{int}\left(\frac{1}{i}\frac{\delta}{\delta\eta}, \frac{1}{i}\frac{\delta}{\delta\bar{\eta}}\right)dx\right] Z_0(\eta,\bar{\eta})$$

展开后第三项：

$$-\frac{1}{2}\int dx\,dy\,dx'\,dy'\,\bar{\eta}(x)S(x-y)\eta(y)\bar{\eta}(x')S(x'-y')\eta(y')$$

loop 贡献了：

$$\frac{\delta^2}{\delta\bar{\eta}_i(z)\delta\eta_j(z)}\frac{\delta^2}{\delta\bar{\eta}_k(z')\delta\eta_l(z')}\underline{Z(\eta,\bar{\eta})}$$

$\Rightarrow + S_{il}(z-z')S_{kj}(z'-z)$