Quantum Yang Mills

2023年5月1日 14:58

1. Recall: How did we quantize electronic field?

In QED we know, Lagrangian for massless spin-1 is $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + J^\mu A_\mu$. so space-time EoM is: $(\Box + m^2)A_\nu = J_\nu \quad (m \to 0)$.

Fourier trans to momentum space we get: $(k^2 g_{\mu\nu} - k_\mu k_\nu)A^\nu = J^\nu$.

While $\det(k^2 g_{\mu\nu} - k_\mu k_\nu) = 0$. directly inversing this matrix is dangerous. Actually, it manifests gauge invariance: $A_\mu \to A_\mu + \partial_\mu \alpha(x)$.

Thus, many different fields, whose differences are just gauge terms $\partial_\mu \alpha(x)$, associate with the same current $J^\mu(x)$.

Recall that $\partial_\mu A^\mu \to \partial_\mu A^\mu + \Box\alpha$. we try to find α: $\Box\alpha = \partial_\mu A^\mu$. express $\alpha(x)$ as: $\alpha(x) = \Box^{-1}(\partial_\mu A^\mu)$.

let $f(\xi) = \int D\pi\, e^{-i\int d^4x \frac{1}{2\xi}(\Box\pi)^2}$ is a function of ξ. shift the field by: $\pi(x) \to \pi(x) - \alpha(x) = \pi(x) - \Box^{-1}(\partial_\mu A^\mu)$.

$\to f(\xi) = \int D\pi\, e^{-i\int d^4x \frac{1}{2\xi}(\Box\pi - \partial_\mu A_\mu)^2}$. because the integral keeps under a shift (something subtle, why local shifts keep integral unchanged?).

Correlation function: $\langle\Omega| T\{O(x_1 \cdots x_n)\}|\Omega\rangle = \frac{1}{Z[0]}\int DA_\mu D\phi_i D\phi_i^*\, e^{i\int d^4x \mathcal{L}[A,\phi]}\, O(x_1 \cdots x_n)$. just require $O(x_1 \cdots x_n)$ to be gauge invariant.

$$= \frac{1}{Z[0]}\frac{1}{f(\xi)}\int D\pi DA_\mu D\phi_i D\phi_i^*\, e^{i\int d^4x\{\mathcal{L}[A,\phi] - \frac{1}{2\xi}(\Box\pi - \partial_\mu A^\mu)^2\}}\, O(x_1 \cdots x_n).$$

now we set $\pi(x)$ as the phase factor in gauge theory: $A_\mu \to A_\mu + \partial_\mu \pi(x)$. $\phi_i \to e^{i\pi(x)}\phi_i$. since $D\pi . DA_\mu . D\phi . O(x_1 \cdots x_n) . \mathcal{L}[A,\phi]$ are gauge invariant. we know the correlation function must equal to that when $\partial\pi(x) = 0$ (gauge fixed).

$$\Rightarrow \langle\Omega| T\{O(x_1 \cdots x_n)\}|\Omega\rangle = \frac{1}{Z[0]}\left(\frac{1}{f(\xi)}\int D\pi\right)\int DA_\mu D\phi_i D\phi_i^*\, e^{i\int d^4x\{\mathcal{L}[A,\phi] - \frac{1}{2\xi}(\partial_\mu A^\mu)^2\}}\, O(x_1 \cdots x_n).$$

It's easy to check: $Z[0] = \left(\frac{1}{f(\xi)}\int D\pi\right)\int DA_\mu D\phi_i D\phi_i^*\, e^{i\int d^4x\{\mathcal{L}[A,\phi] - \frac{1}{2\xi}(\partial_\mu A^\mu)^2\}}$

$$\Rightarrow \langle\Omega| T\{O(x_1 \cdots x_n)\}|\Omega\rangle = \frac{\int DA_\mu D\phi_i D\phi_i^*\, e^{i\int d^4x\{\mathcal{L}[A,\phi] - \frac{1}{2\xi}(\partial_\mu A^\mu)^2\}}\, O(x_1 \cdots x_n)}{\int DA_\mu D\phi_i D\phi_i^*\, e^{i\int d^4x\{\mathcal{L}[A,\phi] - \frac{1}{2\xi}(\partial_\mu A^\mu)^2\}}}$$

Thus, a gauge-fixed lagrangian seems no different from a gauge-invariant lagrangian. So correlation function of fields must be free of ξ.

But this is only valid for gauge invariant correlation functions. $\langle\Omega|A_\mu(x)A_\nu(y)|\Omega\rangle$, or $\langle\Omega|\bar{\psi}(x_1)\psi(x_2)|\Omega\rangle$, e.g. will depend on ξ. However, since S-matrix is gauge invariant, we can cancel all ξ terms in perturbation theory.

2. Faddeev-Popov procedure

For non-Abelian theory, we can try to apply the same method in QED. now we need N^2-1 fields $\pi^a(x)$. implying $\pi^a(x)$ in the adjoint repr.

In YM, we checked for A_μ^a. changes: $A_\mu^a \to A_\mu^a + \frac{1}{g}\partial_\mu\pi^a + f^{abc}A_\mu^b\pi^c$

in adjoint repr. $D_\mu\pi^a = \partial_\mu\pi^a + gf^{abc}A_\mu^b\pi^c$. so $A_\mu^a \to A_\mu^a + \frac{1}{g}D_\mu\pi^a$. (actually, $D_\mu\pi^a$ should be $D_\mu^{ab}\pi^b = \partial_\mu\pi^a + gf^{acb}A_\mu^c\pi^b$. $D_\mu\pi^a$ is simplicity).

We can similarly call $f[A,\xi] = \int D\pi\, \exp[-i\int d^4x \frac{1}{2\xi}(\partial^\mu D_\mu\pi^a)^2]$ and gauge: $\partial_\mu A^{a\mu} = \frac{1}{g}\partial^\mu D_\mu^a\alpha^a[A]$ has a solution.

Similarly shift: $\pi^a \to \pi^a - \frac{1}{g}\alpha^a[A]$. $\Rightarrow f[A,\xi] = \int D\pi\, \exp[-i\int d^4x \frac{1}{2\xi}(\partial_\mu A^{\mu a} - \partial^\mu D_\mu\pi^a)^2]$

why in non-Abelian gauge theory, f must be functional of A?

Fix the gauge and cancel π-terms: $\int DA_\mu D\phi_i\, e^{i\int d^4x \mathcal{L}[A,\phi_i]} = \left(\int D\pi\right)\int DA_\mu D\phi_i \frac{1}{f[A]}\exp\left(i\int d^4x \mathcal{L}[A,\phi_i] - \frac{1}{2\xi}(\partial_\mu A^\mu)^2\right)$

for $f[A]$. we cannot get it out of $\int DA_\mu$.

Here we consider the origin form of $f[A]$: $f[A] = \int D\pi\, \exp(-i\int d^4x \frac{1}{2\xi}(\partial^\mu D_\mu\pi^a)^2)$.

$$= \sqrt{\frac{1}{\det(\partial^\mu D_\mu)^2}} \times \text{const}. \quad \text{(Gauss Integral for quadratic).}$$

so that $Z[0] = \text{const.} \times \int DA_\mu D\phi_i [\det(\partial^\mu D_\mu)]\, \exp\{i\int d^4x[\mathcal{L}(A,\phi_i) - \frac{1}{2\xi}(\partial_\mu A^{a\mu})^2]\}$.

We've known for Grassmann number c & $\bar{c}$, $\det(O) = \int D\bar{c}Dc\, \exp(-i\int d^4x \bar{c}Oc)$ $\Rightarrow$ $\det(\partial^\mu D_\mu) = \int D\bar{c}Dc\, \exp[i\int d^4x\, \bar{c}(-\partial^\mu D_\mu)c]$.

$\Rightarrow Z[0] = \text{const.} \times \int DA_\mu D\phi_i D\bar{c}Dc\, \exp\{i\int d^4x[\mathcal{L}[A,\phi_i] - \frac{1}{2\xi}(\partial_\mu A^{a\mu})^2 - \bar{c}^a\partial^\mu D_\mu c^a]\}$.

c and $\bar{c}$ are two anti-commuting Lorentz scalar, called Faddeev-Popov ghosts and F-P anti-ghosts.

this violate spin-statistics theorem, so ghosts are unphysical.

for gauge boson field ϕ_i. Lagrangian is: $\mathcal{L}_{R\text{-}\xi} = -\frac{1}{4}(F_{\mu\nu}^a F^{a\mu\nu}) - \frac{1}{2\xi}(\partial_\mu A^{a\mu})^2 + (\partial_\mu\bar{c}^a)(D_\mu c^a)$

$= (\partial_\mu\bar{c}^a)(\delta^{ac}\partial_\mu + f^{abc}A_\mu^b)c^c$

gauge boson propagator is: $\nu, b \sim\sim\sim \mu, a$ (momentum p) $= i\delta^{ab}\frac{-g^{\mu\nu} + (1-\xi)\frac{p^\mu p^\nu}{p^2}}{p^2 + i\varepsilon}$ very similar to photon propagator. we can take $\xi = 1$ and cancel $p^\mu p^\nu$ term.

$$\overset{\xi=1}{\sim} \frac{-ig^{\mu\nu}}{p^2 + i\varepsilon}\delta^{ab}.$$

Attention: c and $\bar{c}$ are different Grassmann number, we never expected them to conjugate. They are entirely independent.

3. BRST invariance (a classical symmetry).

We've know that Feynman R-ξ gauge can cancel DoF. A question is, what's the difference between R-ξ gauge and Lorenz/Coulomb gauge?

In QED, we introduce ξ-term: $\frac{1}{2\xi}(\partial^\mu A_\mu)^2$. this partly break the gauge symmetry: $A_\mu(x) \to A_\mu(x) + \partial_\mu\alpha(x)$ cannot hold $\mathcal{L}$ invariant.

However, if we consider some $\alpha(x)$ that $\Box\alpha(x) = 0$, then $\partial^\mu A_\mu(x) \to \partial^\mu A_\mu(x) + \Box\alpha(x) = \partial^\mu A_\mu(x)$. there is still residual symmetry!

we can generally introduce a QED Lagrangian (actually scalar QED) including ghosts: $\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi_i^*)(D^\mu\phi_i) - m^2\phi_i^*\phi_i - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 - \bar{c}\Box c$.

since EoM of $\bar{c}$ and c are just: $\begin{cases}\Box\bar{c} = 0\\ \Box c = 0\end{cases}$. then we can take $\alpha(x) = \theta c(x)$. so that $\alpha(x)$ is a valid gauge holding symmetry.

θ should be a Grassmann number, so that $\alpha(x)$ be a normal number.

Therefore, gauge trans writes as: $\begin{cases}\phi_i(x) \to e^{i\theta c(x)}\phi_i(x) \sim \phi_i(x) + i\theta c(x)\phi_i(x).\\ A_\mu(x) \to A_\mu(x) + \frac{1}{e}\theta\partial_\mu c(x).\end{cases}$

When talking about invariance and symmetry, we have to work under off-shell environment. EoM can just work intuitively, but never used to cancel something.

without $\Box c = 0$ (on-shell EoM), we have $(\partial^\mu A_\mu)^2 \to (\partial^\mu A_\mu)^2 + \frac{2}{e}(\partial^\mu A_\mu)(\theta\Box c) + \frac{1}{e^2}(\theta\Box c)^2$. vanish for $\theta^2 = 0$.

$= (\partial^\mu A_\mu)^2 + \frac{2}{e}(\partial^\mu A_\mu)(\theta\Box c)$

$A_\mu(x) \to A_\mu(x) + \frac{1}{e}\theta\partial_\mu c(x)$.

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$= (\partial^\mu A_\mu)^2 + \frac{2}{e}(\partial^\mu A_\mu)(\theta\Box c)$.

to cancel this extra term, we just let $\bar c(x) \to \bar c(x) - \frac{1}{\xi}\frac{1}{e}\theta\cdot\partial^\mu A_\mu(x)$. then the Lagrangians keep invariant.

This kind of trans is called BRST trans, and the residual symmetry is called BRST s mmetry.

Now back to our Faddeev–Popov Lagrangian: $\mathcal{L}_{FP} = \mathcal{L}_{YM} + \mathcal{L}_{scalars} - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 + (\partial^\mu \bar c^a)(D_\mu c^a)$.

$\sim -\bar c^a\partial^\mu D_\mu c^a$ under IBP.

We still make $\alpha^a(x) = \theta c^a(x)$. then construct BRST trans: $\begin{cases}\phi_i \to \phi_i + i\theta c^a T^a_{ij}\phi_j \\ A^a_\mu \to A^a_\mu + \frac{1}{g}\theta D_\mu c^a\end{cases}$

$$(\partial^\mu A^a_\mu)^2 \to (\partial_\mu A^a_\mu)^2 + \frac{2}{g}\theta(\partial^\mu A^a_\mu)(\partial^\mu D_\mu c^a) \quad\Rightarrow\quad \bar c^a \to \bar c^a - \frac{1}{\xi}\frac{1}{g}\theta(\partial^\mu A^a_\mu).$$

A little difference is that, if we take $-\bar c^a\partial^\mu D_\mu c$. then $D_\mu c$ is not invariant. thus $-\bar c^a\partial^\mu D_\mu c^a \not\to -\bar c^a\partial^\mu D_\mu c^a$, $\bar c$ trans cannot completely cancel these redundant terms.

Now we need to change c^a. so that $D_\mu c^a \to D_\mu c^a$. explicitly:

$$\begin{aligned} D_\mu c^a = (\partial_\mu c^a + gf^{abc}A^b_\mu c^c) &\to \partial_\mu c^a + \partial_\mu\theta\Delta^a + gf^{abc}(A^b_\mu + \tfrac{1}{g}\theta D_\mu c^b)(c^c + \theta\Delta^c) \\ &= \partial_\mu c^a + \partial_\mu\theta\Delta^a + gf^{abc}A^b_\mu c^c + gf^{abc}A^b_\mu\theta\Delta^c + \theta f^{abc}(\partial_\mu c^b + gf^{bde}A^d_\mu c^e)c^c \\ &= \underline{\partial_\mu c^a + gf^{abc}A^b_\mu c^c} + \theta(\partial_\mu\Delta^a + gf^{abc}A^b_\mu\Delta^c) + \theta f^{abc}\partial_\mu c^b c^c + \frac{g\theta}{2}f^{abc}f^{ced}A^b_\mu c^d c^e. \end{aligned}$$

$f^{abc}f^{bde}A^d_\mu c^e c^c = \frac{1}{2}f^{abc}f^{ced}A^b_\mu c^d c^e$

we can read: $\Delta^c = -\frac{1}{2}\theta f^{cab}c^a c^b \quad\Rightarrow\quad c^a \to c^a - \frac{1}{2}\theta f^{abc}c^b c^c$.

These 4 trans construct YM BRST trans. thus $\mathcal{L}_{FP}$ is BRST invariant.

BRST Cohomology:

we try to rewrite BRST trans as an operator $\delta_B: \mathcal{L} \mapsto \Delta\mathcal{L}$ and we get: $\begin{cases}\delta_B\mathcal{L}_{YM} = 0 \\ \delta_B\mathcal{L}_{scalars} = \delta_B\mathcal{L}_{fermions} = 0 \\ \delta_B\mathcal{L}_{gauge} = -\delta_B\mathcal{L}_{ghosts}\end{cases}$ Thus $\delta_B\mathcal{L}_{FP} = 0$.

for a general BRST trans, we expect to have: $U_B = \exp(i\theta Q_B)$. Q_B is the generator of BRST trans, called BRST operator.

since θ is Grassmann. Q_B has to be fermion, and U_B boson. Expand these trans to θ' (for $\theta^2=0$):

$$U_B\phi U_B^{-1} = \phi + i\theta[Q_B, \phi] = \phi + \delta_B\phi.$$
$$U_B\psi U_B^{-1} = \psi + i\theta\{Q_B, \psi\} = \psi + \delta_B\psi.$$

$\cdots$ That's under Dirac picture. we consider how operator trans.

An important property is that $Q_B^2 = 0$. for boson, fermion, and gauge field operator.

$Q_B\cdot Q_B = 0 \Rightarrow \mathrm{Im}\,Q_B \subseteq \ker Q_B$. thus we construct $H(Q_B) = \ker Q_B / \mathrm{Im}\,Q_B$. this is BRST cohomology.

Space of states $\mathcal{F} = \oplus_g \mathcal{F}_g$, g is a integer. thus $Q_B: \mathcal{F}_n \to \mathcal{F}_{n+1}$. Or explicitly: $Q_{Bg}: \mathcal{F}_g \to \mathcal{F}_{g+1}$. $\begin{cases}\ker Q_{Bg} = \{\psi \in \mathcal{F}_g \mid Q_{Bg}\psi = 0\} \\ \mathrm{Im}\,Q_{Bg} = \{Q_{Bg}\psi \mid \psi \in \mathcal{F}_g\}.\end{cases}$

elements in $\ker Q_g$ are BRST cocycle
elements in $\mathrm{Im}\,Q_g$ are BRST coboundary

so $H^g(Q_B) = \frac{\ker Q_{Bg}}{\mathrm{Im}\,Q_{Bg-1}}$. $H(Q_B) = \oplus_g H^g(Q_B)$. that's a natural sheaf structure. g is the 'ghosts number'.

Physical states are always no ghosts. so $H^0(Q_B) = \frac{\ker Q_{B0}}{\mathrm{Im}\,Q_{B-1}}$ are space of gauge invariant quantites.

4. Feynman Rules for YM Theory:

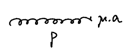
Now we completely write down the SU(N) invariant Lagrangian with non-Abelian gauge-fields:

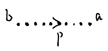
$$\mathcal{L} = \underbrace{-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}}_{\text{non-Abelian fields}} \underbrace{- \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2}_{\text{R-}\xi\text{ gauge term}} + \underbrace{(\partial^\mu\bar c^a)(\delta^{ac}\partial_\mu + f^{abc}A^b_\mu)c^c}_{\text{Faddeev-Popov ghosts}} + \bar\psi_i(i\not\partial - m)\psi_i + \underbrace{\bar\psi_i g\not A^a T^a_{ij}\psi_j}_{\text{Fermion-gauge interaction}} + \underbrace{[(\delta_{ki}\partial^\mu - igA^{a\mu}T^a_{ki})\phi_i]^*[(\delta_{kj}\partial_\mu - igA^a_\mu T^a_{kj})\phi_j]}_{\text{including boson-gauge interaction}} - M^2\phi_i^*\phi_i.$$

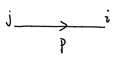
$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc}A^b_\mu A^c_\nu$.

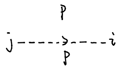
Kinetic terms: $\mathcal{L}_{kin} = -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 + \bar\psi_i(i\not\partial - m)\psi_i - \phi_i^*(\Box + M^2)\phi_i - \bar c^a\Box c^a$.

$\Rightarrow$ propagator:

 $= \frac{-ig^{\mu\nu}}{p^2+i\varepsilon}\delta^{ab}$ $(\xi=1)$. gauge boson propagator (e.g. gluons),

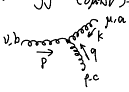
 $= \frac{i\delta^{ab}}{p^2+i\varepsilon}$ ghost propagators

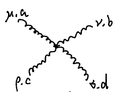
 $= \frac{i\delta^{ij}}{\not p - m + i\varepsilon}$ colored fermions propagators.

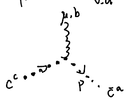
 $= \frac{i\delta^{ij}}{p^2 - M^2 + i\varepsilon}$ colored bosons propagators

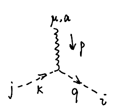
Interaction terms: $\mathcal{L}_{int} = -gf^{abc}(\partial_\mu A^a_\nu)A^{b\mu}A^{c\nu} - \frac{1}{4}g^2(f^{eab}A^{a\mu}A^{b\nu})(f^{ecd}A^c_\mu A^d_\nu) + gf^{abc}(\partial^\mu\bar c^a)A^b_\mu c^c + gA^a_\mu\bar\psi_i\gamma^\mu T^a_{ij}\psi_j + igA^{a\mu}T^a_{ij}(\phi_i^*\partial_\mu\phi_j - \phi_j\partial_\mu\phi_i^*) + g^2\phi_i^*A^a_\mu T^a_{ik}T^b_{kj}A^b_\mu\phi_j$.

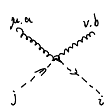
$\Rightarrow$ Vertex:

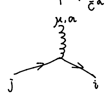
 $= gf^{abc}[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\nu + g^{\rho\mu}(q-k)^\nu]$. $p+k+q=0$.

 $= -ig^2[f^{abe}f^{cde}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})]$

 $= -gf^{abc}p^\mu$

 $= ig(k^\mu + q^\mu)T^a_{ij}$.

 $= ig^2 T^a_{ik}T^b_{kj}g^{\mu\nu}$.

 $= ig\gamma^\mu T^a_{ij}$.