Ward - Takahashi identity

Schwinger - Dyson $\quad \tilde{\varphi}_a(x) = \varphi_a(x) + \delta\varphi_a(x)$

$$Z[J] = \int D\phi\, e^{i[S[\phi] + \int d^4x\, J\phi]} \qquad D\tilde{\varphi} = D\varphi$$

$$0 = \delta Z[J] = \int D\phi\, e^{iS + i\int d^4x\, J\phi}\; i\int d^4x \left[\frac{\delta S}{\delta\varphi_a(x)} + J_a(x)\right]\delta\varphi_a(x)$$

$$\frac{\delta S}{\delta\varphi_a(x)} = \frac{\partial\mathcal{L}}{\partial\varphi_a} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_a)}, \qquad \delta\phi = F[\phi]\,\epsilon$$

$$-i\frac{\delta}{\delta J_{a_1}(x_1)} \cdots -i\frac{\delta}{\delta J_{a_n}(x_n)}, \quad \text{then } J = 0$$

$$\left\langle \frac{\delta S}{\delta\varphi_a(x)} F[\hat{\varphi}]\, \hat{\varphi}_1 \cdots \hat{\varphi}_n \right\rangle = i\sum_{j=1}^{n} \langle \varphi_1 \cdots \varphi_{j-1}\, \delta_{aa_j}\, \delta^4(x-x_j)\, \varphi_{j+1} \cdots \varphi_n \rangle$$

$$\partial_\mu J^\mu_k = \frac{\delta S}{\delta\varphi_a(x)} F_{ak}[\varphi]$$

$$\partial_{x\mu} \langle \hat{J}^\mu_k(x)\, \varphi_1 \cdots \varphi_n \rangle = i\sum_{j=1}^{n} \langle \varphi_1 \cdots \varphi_{j-1}\, \delta^4(x-x_j)\, F_{a_jk}[\varphi](x_j)\, \varphi_{j+1} \cdots \varphi_n \rangle$$

$$[\hat{Q}_k, \hat{\varphi}_j] = i F_{a_jk}[\hat{\varphi}](x_j) \qquad \psi\gamma^\mu\bar{\psi}$$

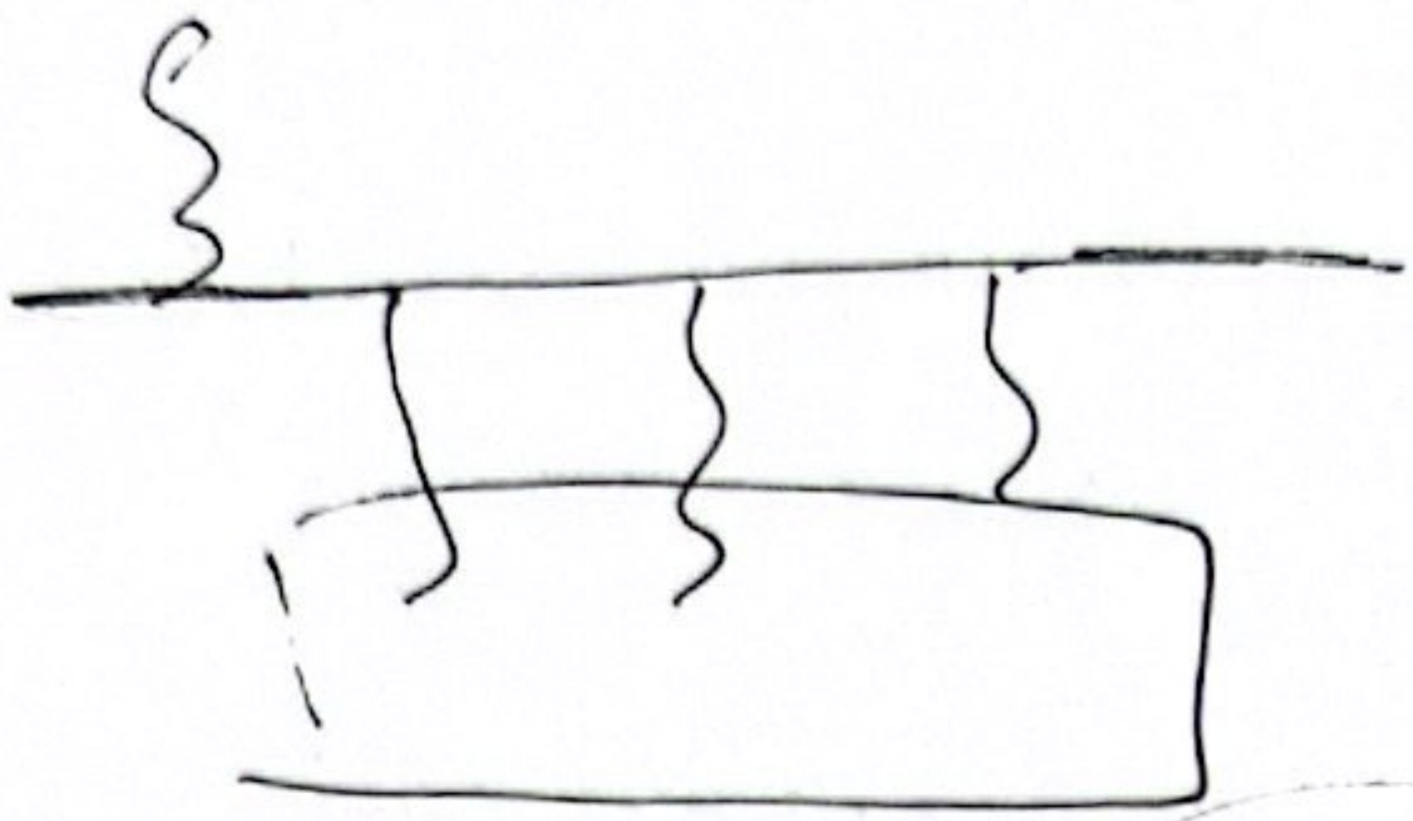
In QED, $\mathcal{L}_\xi \quad \frac{1}{2\xi}(\partial\cdot A)^2, \quad \psi \to e^{i\phi}\psi, \quad [Q, \hat{\varphi}] = -e\bar{\psi}$

$$\partial_{x\mu} \langle \psi\gamma^\mu\bar{\psi}\, \psi(x_1)\psi(x_2) \rangle = [-e\,\delta^4(x-x_1) + e\,\delta^4(x-x_2)] \langle \psi(x_1)\bar{\psi}(x_2) \rangle$$

$x_1 \to p, \quad x_2 \to q, \quad x \to k$

k_μ 

$$k_\mu \mathcal{M}_3^\mu(k, p, q) = -e\mathcal{M}_2(p, q-k) + e\mathcal{M}_2(p+k, q)$$



$k_\mu^{\pi} \mathcal{M}^{\mu} = 0$

$$-g^{\mu\nu} - \cancel{\left(1-\frac{1}{\xi}\right)} \frac{p^\mu p^\nu}{p^2}$$

p^ν

$$\sum_{\lambda=0}^{3} \varepsilon_\mu^{(\lambda)} \varepsilon_\nu^{(\lambda)} \Rightarrow -g_{\mu\nu}$$

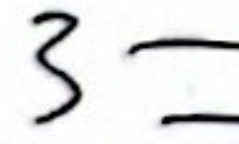
$$\sum_{\lambda=1}^{2} \varepsilon_\mu^{(\lambda)} \varepsilon_\nu^{(\lambda)} = -g^{\mu\nu} + \frac{p^\mu p^\nu}{\cdots}$$

$$\mathcal{L} \doteq \cdots + m^2 A_\mu A^\mu$$

Complete propagator & vertex function $\to -g^{\mu\nu}$

$$\tau(x_1, \cdots x_n) = G^{(n)}(x_1, \cdots, x_n)$$
$$= \frac{1}{i^n} \frac{\delta^n Z[J]}{\delta J(x_1) \cdots \delta J(x_n)}\bigg|_{J=0}$$

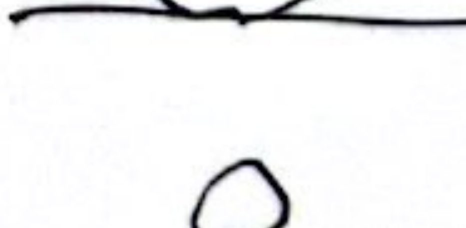
$W = -i \ln Z$, $\quad i\phi(x_1, \cdots, x_n) = G_c^{(n)}(x_1, \cdots, x_n) = \frac{1}{i^{n-1}} \frac{\delta^n W}{\delta \cdots J}\Big|_{J=0}$

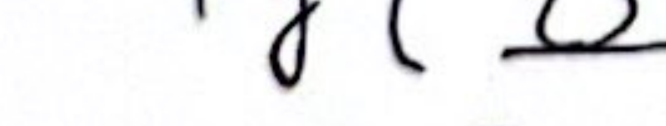
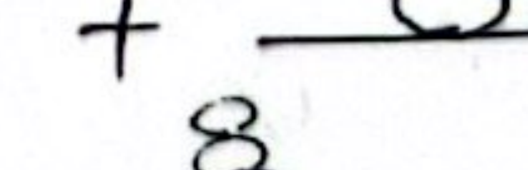
$G^{(4)} = -3$  $-3ig$  $- ig \times + O(g^2)$

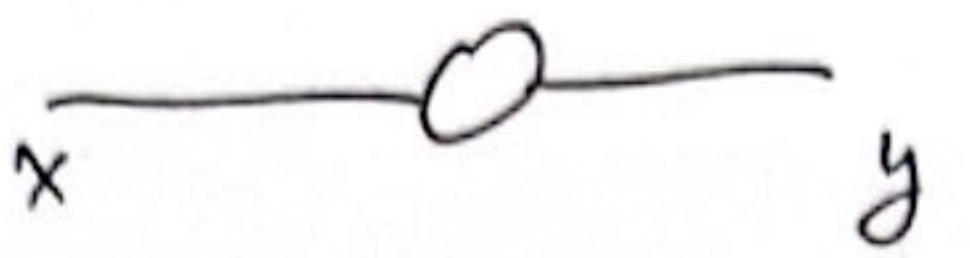
$$G_c^{(4)} = -ig \times + O(g^2)$$

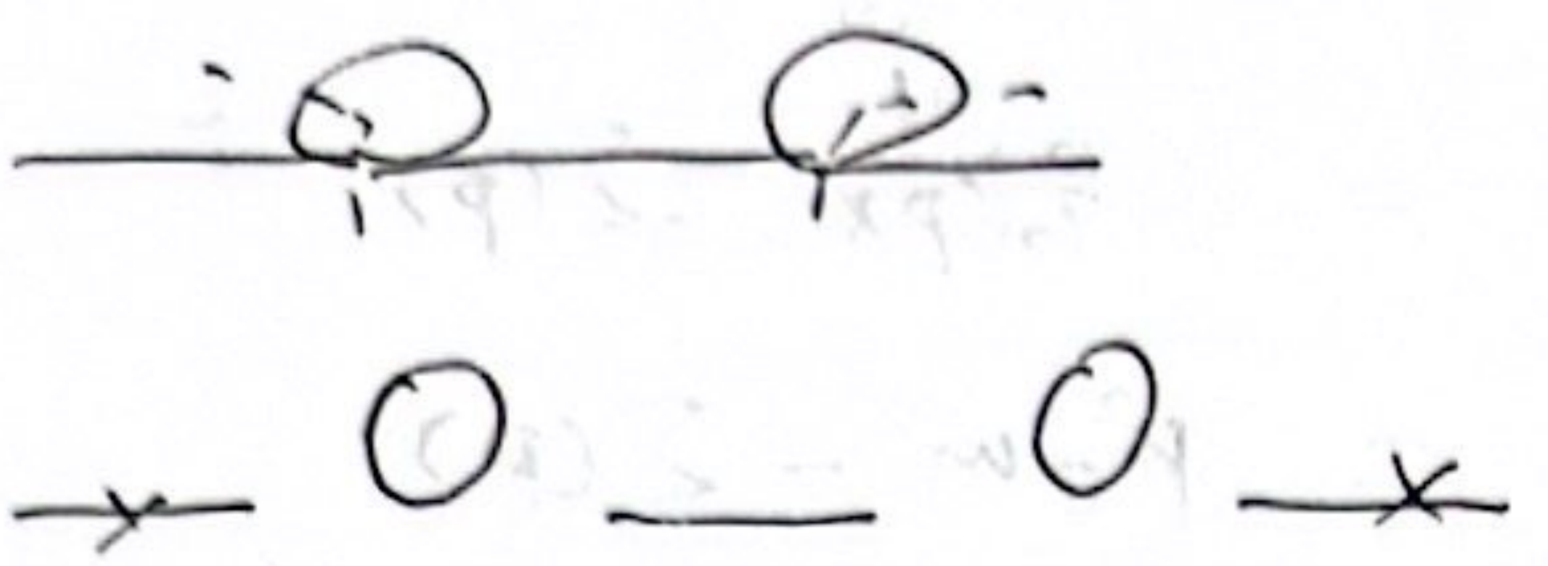
$$G^{(n)} = \sum_{m+l=n} G_c^{(m)} G_c^{(l)}$$

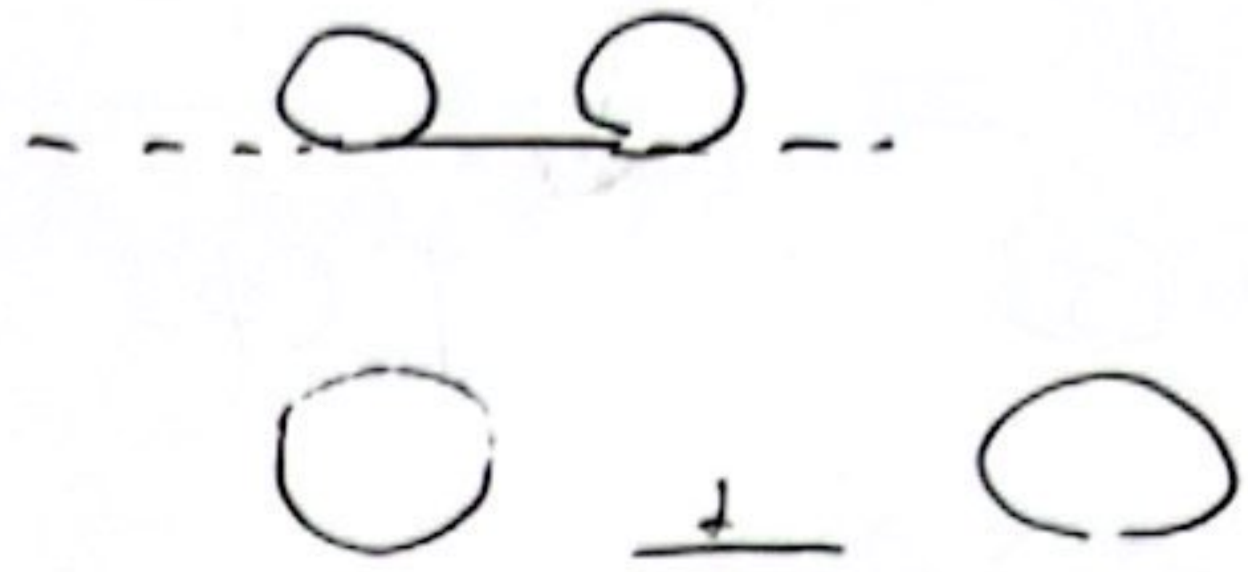
~~$G^{(2)}$~~ $\quad G_c^{(2)} =$  $+g$ 

$+g^2 [$ $+$  ~~]~~ $+$ $]$

$+g^3 [$  $+$  $+ \cdots] + O(g^4)$

$G_c^{(2)}(x-y) =$ 

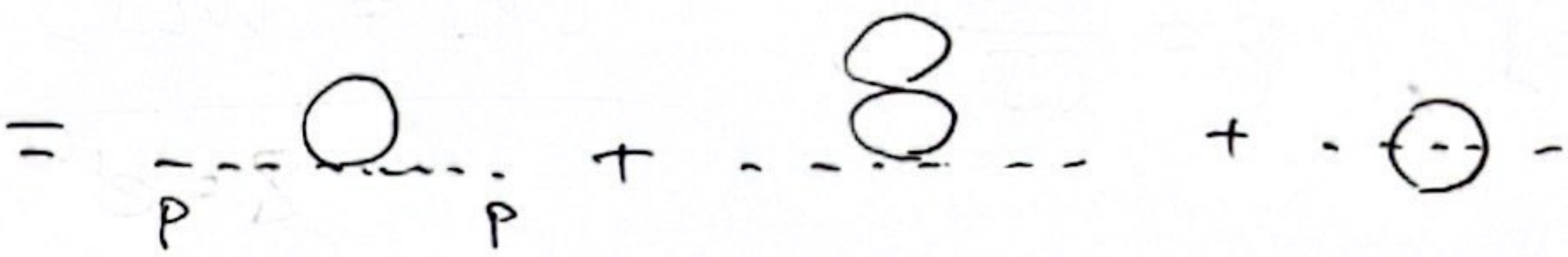




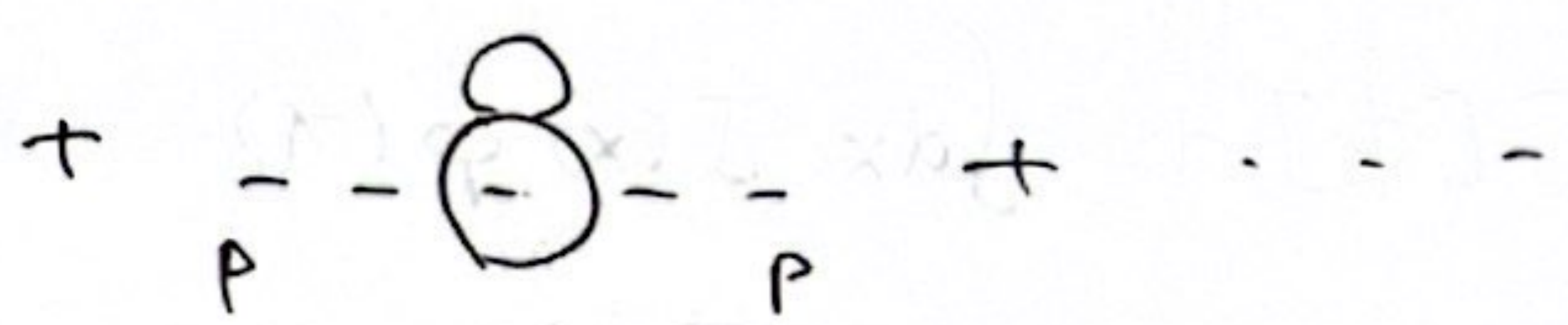
1 - particle reducible

1PI graph.

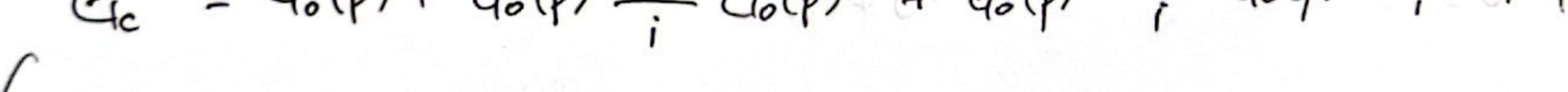
 $= \frac{1}{i}\Sigma(p)$



$G_0(p) = \frac{i}{p^2 - m^2}$



$$G_c^{(2)} = G_0(p) + G_0(p)\frac{\Sigma(p)}{i}G_0(p) + G_0(p)\frac{\Sigma(p)}{i}G_0(p)\frac{\Sigma(p)}{i}G_0(p) + \cdots$$



$$= \frac{G_0(p)}{1 - \frac{\Sigma}{i}G_0} = \left[G_0^{-1} - \frac{1}{i}\Sigma(p)\right]^{-1} = \frac{i}{p^2 - m^2 - \Sigma(p)}$$

$$m_{phy}^2 = m^2 + \Sigma(p)$$

2-point vertex function

$$G_c^{(2)}(p)\,\Gamma^{(2)}(p) = i \quad \Rightarrow \quad \Gamma^{(2)}(p) = i\, G_c^{(2)}(p)^{-1}$$

$$\downarrow \qquad\qquad \downarrow$$

$$W[J] \qquad\qquad \Gamma[\phi]$$

$$= i\, G_0^{-1}(p) - \Sigma(p)$$

$$= p^2 - m^2 - \Sigma(p)$$

$$f(x), \quad u = \frac{df}{dx}, \quad g = f - ux, \quad g(u)$$

$$U(S,V), \quad T = \left(\frac{\partial U}{\partial S}\right)_V,$$

$$Z = e^{-\beta F} \qquad F(T,V) = U - TS$$

$$L(x,\dot{x}), \quad p = \frac{\partial L}{\partial \dot{x}}, \quad H = \cancel{L} - \dot{x}p$$

$$W[J], \quad \phi = \frac{\delta W}{\delta J}, \quad \Gamma[\phi] = W[J] - \int J\phi$$

$$Z = e^{iW}$$

$$W[J] = \Gamma[\phi] + \int dx\, J(x)\,\phi(x)$$

$$\frac{\delta W}{\delta J(x)} = \phi(x), \qquad \frac{\delta \Gamma[\phi]}{\delta \phi(x)} = -J(x)$$

$$G_c(x,y) = -i\,\frac{\delta^2 W}{\delta J(x)\,\delta J(y)} = -i\,\frac{\delta \phi(x)}{\delta J(y)}$$

$$\Gamma(x,y) = \frac{\delta \Gamma[\phi]}{\delta \phi(x)\,\delta \phi(y)} = -\frac{\delta J(x)}{\delta \phi(y)}$$

$$\int dz\, G_c(x,z)\,\Gamma(z,y) = i\,\delta(x-y)$$

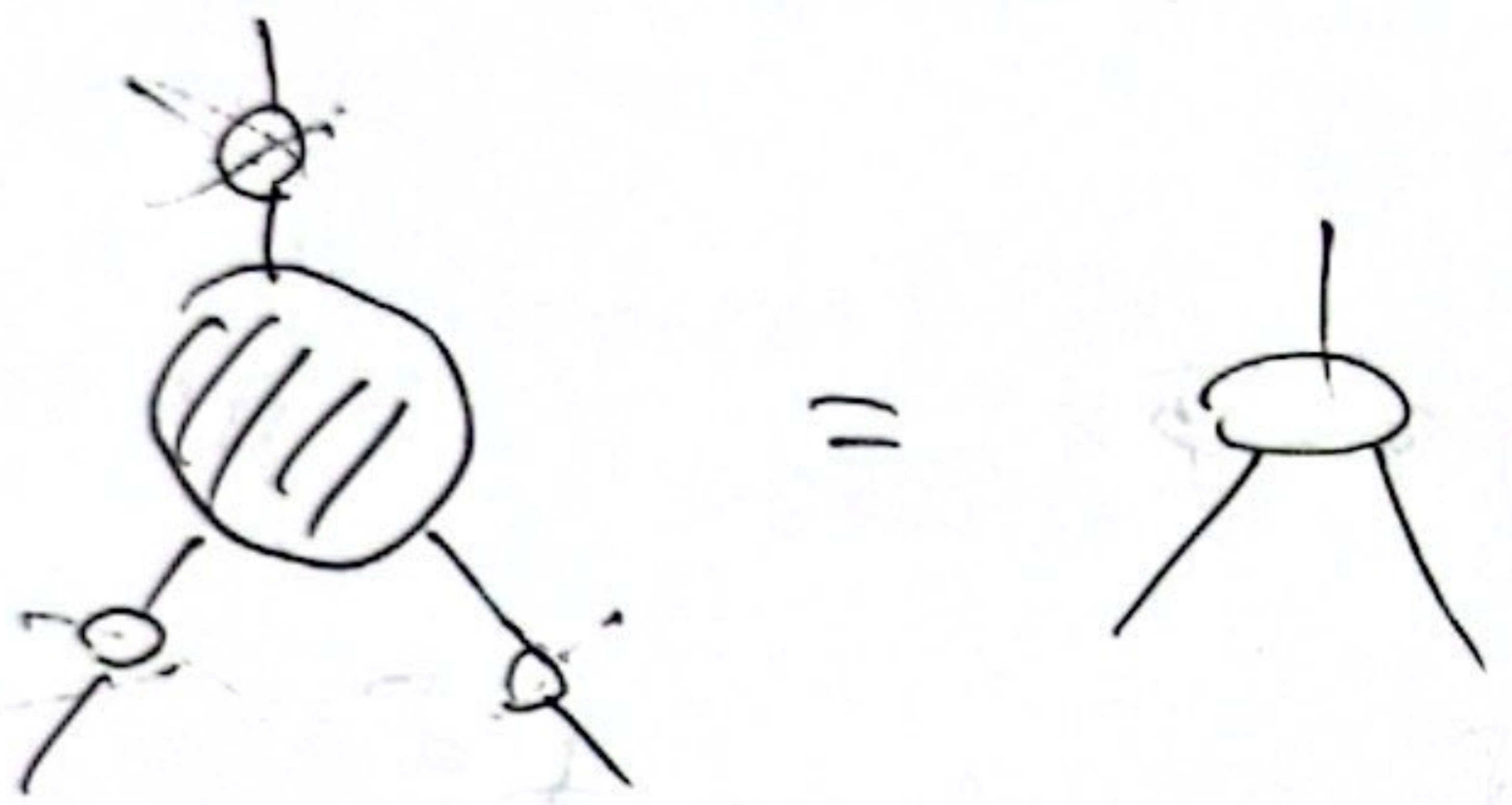
$$\frac{\delta}{\delta J(\ldots}$$

$$\int dz \frac{\delta^3 W[J]}{\cancel{\delta J(x)\delta}\ \delta J(x)\ \delta J(x'')\ \delta J(z)} \frac{\delta^2 \Gamma}{\delta\phi(z)\delta\phi(y)}$$

$$+ \frac{\delta^2 W}{\delta J(x)\ \delta J(z)} G_c(y',x'') \frac{\delta^3 \Gamma}{\delta\phi(z)\delta\phi(y)\delta\phi(y')} = 0.$$

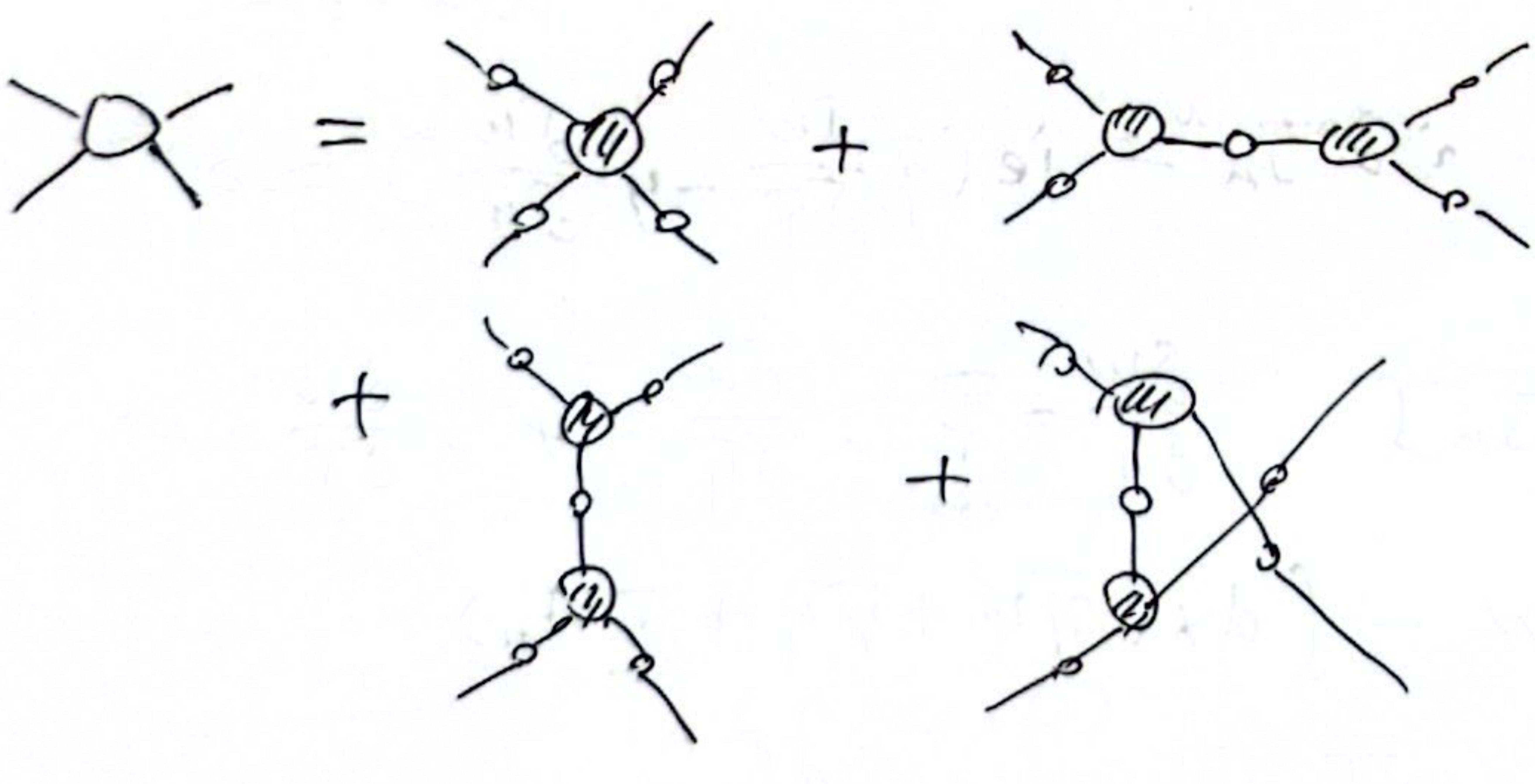
$$\frac{\delta}{\delta J(x'')} = i\int dy\, G_c(y;x'') \frac{\delta}{\delta\phi(y)}$$

$$i \frac{\delta^3 W}{\delta J(x)\ \delta J(x')\ \delta J(x'')} = \int dz\, dz'\, dz''\, G_c(x,z) G_c(x',z') G_c(x'',z'') \frac{\delta^3 \Gamma}{\delta\phi(z)\delta\phi(z')\delta\phi(z'')}$$



1PI

$$\boxed{\frac{\delta^3 \Gamma}{\delta\phi(y)\delta\phi(y')\delta\phi(y'')}} = -\int dx\, dx'\, dx''\, \Gamma(x,y)\Gamma(x',y')\Gamma(x'',y'') \frac{\delta^3 W}{\delta J(x)\delta J(x')\delta J(x'')}$$



QED

$$\mathcal{L}_\xi = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar\psi(i\not\partial - m)\psi - \frac{1}{2\xi}(\partial\cdot A)^2 + J^\mu A_\mu + \bar\eta\psi + \bar\psi\eta$$

$+ \cdots$

$$Z = N\int DA_\mu\, D\bar\psi\, D\psi\, \exp(i\int \mathcal{L}_\xi\, dx).$$

$$A_\mu \to A_\mu + \partial_\mu\Lambda \qquad \psi \to \psi - ie\Lambda\psi \qquad \bar\psi \to \bar\psi + ie\Lambda\bar\psi$$

$$\exp\{i\int dx\,[-\frac{1}{\xi}(\partial\cdot A)\Box\Lambda + J^\mu\partial_\mu\Lambda - ie\Lambda(\bar\eta\psi - \bar\psi\eta)]\}$$

$$= 1 + i\int dx\,[\;\cdots\;]\;\Lambda$$

$$-\frac{1}{\xi}\Box(\partial\cdot A) - \partial^\mu J_\mu - ie(\bar\eta\psi - \bar\psi\eta)$$

$$\psi \to \frac{1}{i}\frac{\delta}{\delta\bar\eta}, \quad \bar\psi \to \frac{1}{i}\frac{\delta}{\delta\eta}, \quad A_\mu \to \frac{1}{i}\frac{\delta}{\delta J}$$

$$\left[\frac{i}{\xi}\Box\partial^\mu\frac{\delta}{\delta J^\mu} - \partial^\mu J_\mu - e\left(\bar\eta\frac{\delta}{\delta\bar\eta} - \eta\frac{\delta}{\delta\eta}\right)\right] Z[\eta,\bar\eta,J] = 0$$

$$W = -i\ln Z$$

$$-\frac{\Box}{\xi}\partial^\mu\frac{\delta W}{\delta J^\mu} - i\partial^\mu J_\mu - ie\left(\bar\eta\frac{\delta W}{\delta\bar\eta} - \eta\frac{\delta W}{\delta\eta}\right) = 0$$

$$W[\eta,\bar\eta,J_\mu] \qquad \frac{\delta W}{\delta\eta} = \bar\psi, \quad \frac{\delta W}{\delta\bar\eta} = \psi, \quad \frac{\delta W}{\delta J_\mu} = A^\mu$$

$$\Gamma[\psi,\bar\psi,A_\mu] = W - \int dx(\bar\eta\psi + \bar\psi\eta + J^\mu A_\mu)$$

$$\frac{\delta\Gamma}{\delta A_\mu} = -J^\mu, \quad \frac{\delta\Gamma}{\delta\psi} = -\bar\eta, \quad \frac{\delta\Gamma}{\delta\bar\psi} = -\eta$$

$$\cancel{-\frac{1}{\xi}\partial^\mu A_\mu} + i\,\partial_\mu \frac{\delta\Gamma}{\delta A_\mu} + ie\psi\frac{\delta\Gamma}{\delta\psi} - ie\bar\psi\frac{\delta\Gamma}{\delta\bar\psi} = 0$$

$$\frac{\delta}{\delta\bar\psi(x_1)}\frac{\delta}{\delta\psi(y_1)} \qquad \bar\psi = \psi = A_\mu = 0$$

$$i\,\partial_x^\mu \frac{\delta^3\Gamma[0]}{\delta\bar\psi(x_1)\,\delta\psi(y_1)\,\delta A^\mu(x)} = ie\frac{\delta\Gamma[0]}{\delta\bar\psi(x_1)\,\delta\psi(y_1)}\delta(x-x_1) - ie\frac{\delta^2\Gamma[0]}{\delta\bar\psi(x_1)\,\delta\psi(y_1)}\delta(x-y_1)$$

Define :

$$\int dx\,dx_1\,dy_1\, e^{i(p'x_1-py_1-qx)}\frac{\delta^3\Gamma[0]}{\delta\bar\psi(x_1)\delta\psi(y_1)\delta A^\mu(x)} = ie\,(2\pi)^4\delta(p'-p-q)\,\Gamma_\mu(p,q,p')$$

$$\int dx_1\,dy_1\, e^{i(p'x-py_1)}\frac{\delta^2\Gamma[0]}{\delta\bar\psi(x_1)\delta\psi(y_1)} = (2\pi)^4\delta^4(p'-p)\,S_F'^{-1}(p) \qquad p' = p+q$$

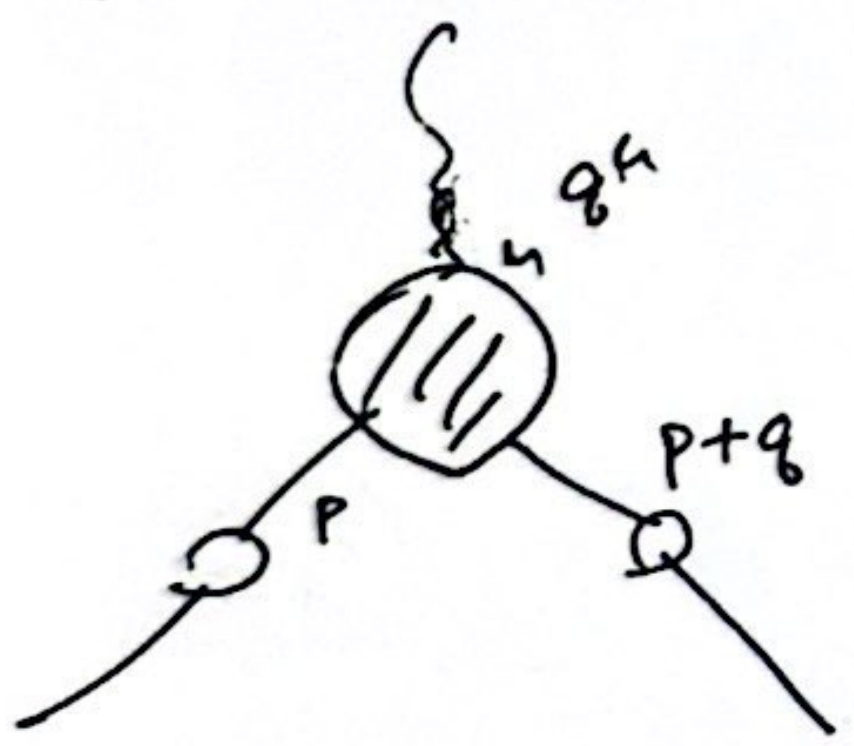
$\cancel{\Gamma(p)\,p^\mu}$

$$q^\mu\Gamma_\mu(p,q,p+q) = S_F'^{-1}(p+q) - S_F'^{-1}(p)$$

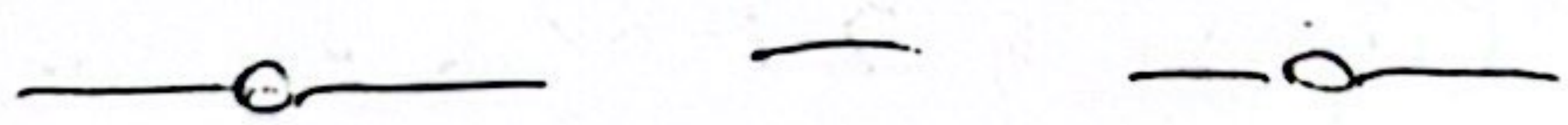
gauge

1PI

$S_F'(p+q)\ S_F'(p)$



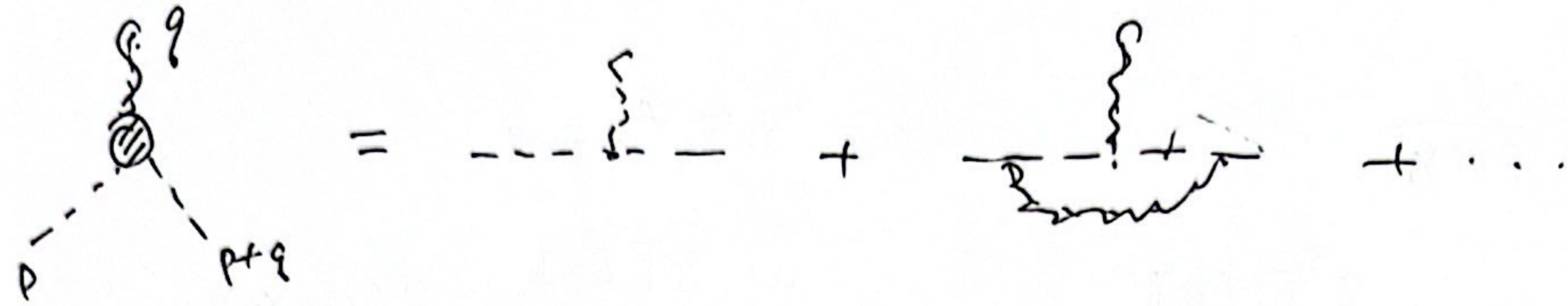
$$= S_F'(p) - S_F(p+q)$$



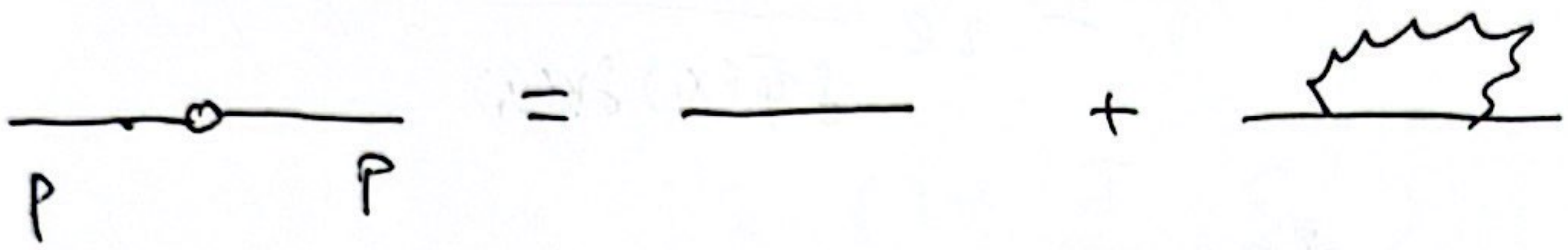
$q^\mu \to 0$ Ward - identity

$$\frac{\partial S_F'^{-1}}{\partial p^\mu} = \Gamma_\mu(p,0,p)$$

$\Gamma_\mu(p, q, p+q)$



S_F'



lowest order

$$S_F^{-1} = \not p - m \quad , \quad \frac{\partial S_F'^{-1}}{\partial p^\mu} = \gamma_\mu$$

$$\Gamma_\mu = \gamma_\mu + \cdots$$

$$\frac{\delta^3 \Gamma}{\delta \bar\psi(x_1)\, \delta\psi(y_1)\, \delta A^\mu} = S_F^{-1}\, S_F^{-1}\, D_{\mu\nu}^{-1}(u-x)\, \frac{\delta W}{\delta\eta\, \delta\bar\eta\, \delta J}$$

$$Z = \exp\left(\frac{1}{i} \frac{\delta}{\partial\eta} \gamma^\mu \frac{\delta}{\delta\bar\eta} \frac{\delta}{\delta J^\mu} \right) Z_0$$

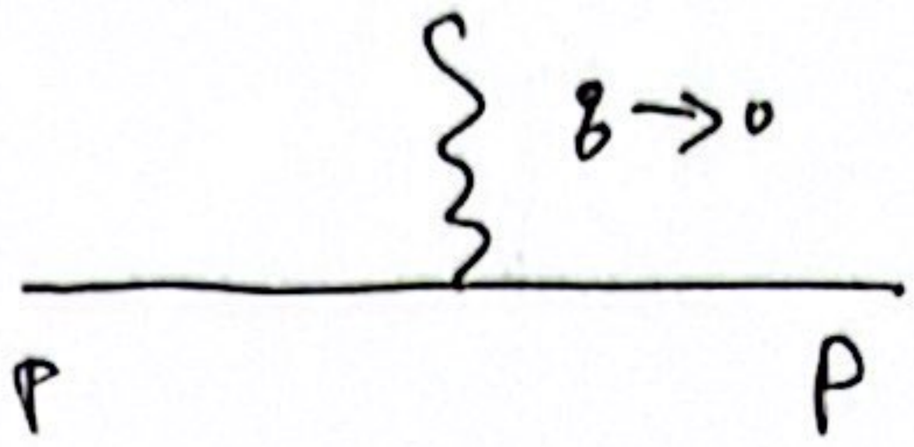
$\bar\psi \gamma^\mu \psi A_\mu$

$$Z_0 = \exp(\bar\eta S_F \eta) \exp(J^\mu D_{\mu\nu} J^\nu) .$$

$$S_F(p)\, S^{-1}(p) = 1$$

$$\frac{\partial S_F}{\partial p} S_F^{-1}(p) + S_F(p) \frac{\partial S_F^{-1}}{\partial p^\mu} = 0$$

$$\Rightarrow \quad \frac{\partial S_F}{\partial p^\mu} = - S_F(p)\, \gamma_\mu\, S_F(p)$$



NLO

$$i S_F' = i S_F + i S_F \frac{\Sigma}{i} i S_F + \cdots$$

$$= i S_F \left(1 + \frac{\Sigma}{i} i S_F'\right) \qquad \Rightarrow S_F'^{-1} = S_F^{-1} - \Sigma$$

$$\frac{\partial S_F'^{-1}}{\partial p^\mu} = \frac{\partial S_F^{-1}}{\partial p^\mu} - \frac{\partial \Sigma}{\partial p^\mu} = \gamma_\mu - \frac{\partial \Sigma}{\partial p^\mu}$$

$$\parallel$$

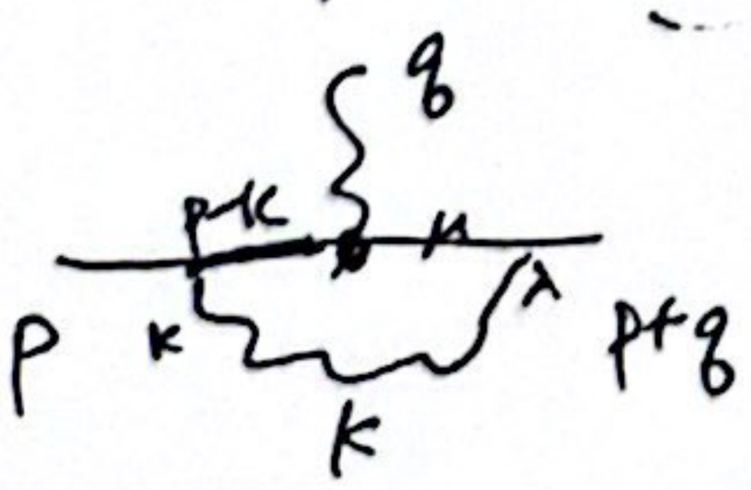
$$\Gamma_\mu(p, q, p+q) = \gamma_\mu + \Lambda_\mu(p, q, p+q)$$

To prove : $\quad \Lambda_\mu(p, 0, p) = - \dfrac{\partial \Sigma}{\partial p^\mu}$

$$\frac{\Sigma}{i} = (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \frac{-i g_{\mu\nu}}{k^2} \gamma^\mu \frac{i}{\not p - \not k - m} \gamma^\nu$$

$$= -e^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \gamma^\mu S_F(p-k) \gamma_\mu$$

$$\frac{\partial \Sigma}{\partial p^\mu} = i e^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \gamma^\nu S_F(p-k) \gamma_\mu S_F(p-k) \gamma_\nu$$



$$-ie \Lambda_\mu(p, q, p+q)$$

$$= (-ie)^3 \int \frac{d^4k}{(2\pi)^4} \frac{-i g_{\kappa\lambda}}{k^2} \gamma^\kappa \frac{i}{\not p - \not k - m} \gamma_\mu \frac{i}{\not p + \not q - \not k - m} \gamma^\lambda$$

$S_F' \to Z_2 S_F$

$\Gamma_\mu(p,0,p) \to \frac{1}{Z_1}\gamma_\mu \quad \rightsquigarrow \quad Z_1 = Z_2$

U(1) ~~SU(N)~~