Fermions

费米子场：反对易：$\{\psi(x),\psi(y)\}\big|_{x^0=y^0}=0$

C-number

Grassmann 生成元 ε_i

$$\{\varepsilon_i,\varepsilon_j\}=0$$

其中 i,j 遍及 $i,j=1,2,\cdots,n$

$$\boxed{\varepsilon_i^2=0}$$

$$f(\varepsilon_1,\varepsilon_2)=a_0+a_1\varepsilon_1+a_2\varepsilon_2+a_3\varepsilon_1\varepsilon_2$$
$$=a_0+a_1\varepsilon_1+a_2\varepsilon_2-a_3\varepsilon_2\varepsilon_1$$

$$\varepsilon_i^2\varepsilon_j=0$$

定义左微分

$$\frac{\partial f}{\partial\varepsilon_1}=\frac{\partial^L f}{\partial\varepsilon_1}=a_1+a_3\varepsilon_2$$
$$\frac{\partial f}{\partial\varepsilon_2}=\frac{\partial^L f}{\partial\varepsilon_2}=a_2-a_3\varepsilon_1$$

同样也可定义右微分

$$\frac{\partial^R f}{\partial\varepsilon_1}=a_1-a_3\varepsilon_2$$

注意到

$$\varepsilon_1\frac{\partial f}{\partial\varepsilon_1}=a_1\varepsilon_1+a_3\varepsilon_1\varepsilon_2$$

$$\varepsilon_1 f=a_0\varepsilon_1+a_2\varepsilon_1\varepsilon_2$$

$$\frac{\partial}{\partial\varepsilon_1}(\varepsilon_1 f)=a_0+a_2\varepsilon_2$$

$$\Rightarrow\ \left(\varepsilon_1\frac{\partial}{\partial\varepsilon_1}+\frac{\partial}{\partial\varepsilon_1}\varepsilon_1\right)f=f$$

或写成

$$\varepsilon_1\frac{\partial}{\partial\varepsilon_1}+\frac{\partial}{\partial\varepsilon_1}\varepsilon_1=1$$

一般地，

$$\{\varepsilon_i, \frac{\partial}{\partial \varepsilon_j}\} = \delta_{ij} \qquad \{\frac{\partial}{\partial \varepsilon_i}, \frac{\partial}{\partial \varepsilon_j}\} = 0$$

$d\varepsilon_i$ 也是 Grassmann 量.

$$\begin{cases} \{\varepsilon_i, d\varepsilon_i\} = 0 \\ \{d\varepsilon_i, d\varepsilon_j\} = 0 \end{cases}$$

$$\int d\varepsilon_1 d\varepsilon_2 f(\varepsilon_1, \varepsilon_2) = \int d\varepsilon_1 \left[\int d\varepsilon_2 f(\varepsilon_1, \varepsilon_2)\right]$$

$\int d\varepsilon_1$, $\int \varepsilon d\varepsilon_1$ = ?

$$\begin{aligned}(\int d\varepsilon_1)^2 &= \int d\varepsilon_1 \int d\varepsilon_2 \\ &= \int d\varepsilon_1 d\varepsilon_2 \\ &= -\int d\varepsilon_2 d\varepsilon_1 \\ &= -(\int d\varepsilon_1)^2\end{aligned}$$

因此 . $\int d\varepsilon_1 = \int d\varepsilon_2 = 0$

定义 $\int d\varepsilon_1 \varepsilon_1 = 1$

即 $\underbrace{\int d\varepsilon_i \varepsilon_i}_{\text{无求和}} = 1 \quad \int d\varepsilon_i = 0$

$f(\varepsilon_1, \varepsilon_2)$ ，代入 $\int d\varepsilon_1 f$

得

$$\begin{aligned}\int d\varepsilon_1 f &= \int d\varepsilon_1 \left[a_0 + a_1\varepsilon_1 + a_2\varepsilon_2 + a_3\varepsilon_1\varepsilon_2\right] \\ &= a_0 \int d\varepsilon_1 + a_1 \int d\varepsilon_1 \varepsilon_1 - a_2\varepsilon_2 \int d\varepsilon_1 + a_3\varepsilon_2 \int d\varepsilon_1 \varepsilon_1 \\ &= a_1 + a_3\varepsilon_2\end{aligned}$$

现在 讨论 η 和 $\bar{\eta}$ 作为相互独立(复) Grassmann 量，因此

$$\int d\eta = \int d\bar{\eta} = 0 \qquad \int d\eta\,\eta = \int d\bar{\eta}\,\bar{\eta} = 1$$

由于 $\eta^2 = \bar{\eta}^2 = 0$

$$e^{\bar{\eta}\eta} = 1 - \bar{\eta}\eta$$

因此

$$\int d\bar{\eta}\, d\eta\, e^{-\bar{\eta}\eta} = \underline{\int d\bar{\eta}\, d\eta} - \int d\bar{\eta}\, d\eta\, \bar{\eta}\eta$$

$$= 0 + \int d\bar{\eta}\, d\eta\, \eta\bar{\eta}$$

$$= 1$$

推广更高维度：2维

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \qquad \bar{\eta} = \begin{pmatrix} \bar{\eta}_1 \\ \bar{\eta}_2 \end{pmatrix}$$

$$\bar{\eta}\eta = (\bar{\eta}^T \eta) = \bar{\eta}_1\eta_1 + \bar{\eta}_2\eta$$

$$\underline{(\bar{\eta}\eta)^2} = (\bar{\eta}_1\eta_1 + \bar{\eta}_2\eta_2)(\bar{\eta}_1\eta_1 + \bar{\eta}_2\eta_2)$$

$$= 2\bar{\eta}\eta_1\bar{\eta}_2\eta_2$$

$$e^{-\bar{\eta}\eta} = 1 - (\bar{\eta}_1\eta_1 + \bar{\eta}_2\eta_2) + \bar{\eta}_1\eta_1\bar{\eta}_2\eta_2$$

代入积分规则，并定义 $d\bar{\eta}\, d\eta = d\bar{\eta}_1\, d\eta_1 \cdot d\bar{\eta}_2 \cdot d\bar{\eta}_2$

$$\int d\bar{\eta}\, d\eta\, e^{-\bar{\eta}\eta} = \int d\bar{\eta}_1\, d\bar{\eta}_2\, d\eta_1\, d\eta_2\, \bar{\eta}_1\eta_1\bar{\eta}_2\eta_2$$

$$= 1$$

做变量变换 $\eta = \underline{M}\alpha \qquad \bar{\eta} = \underline{N}\bar{\alpha}$

$$\eta_1\eta_2 = (M_{11}\alpha_1 + M_{12}\alpha_2)(M_{21}\alpha_1 + M_{22}\alpha_2)$$

$$= (M_{11}M_{22} - M_{12}M_{21})\alpha_1\alpha_2$$

$$= (\det M)\alpha_1\alpha_2$$

但，若让 $\int d\eta_1\, d\eta_2\, \eta_1\eta_2 = \int d\alpha_1\, d\alpha_2\, \alpha_1\alpha_2$

我们必须 $\underline{d\eta_1\, d\eta_2 = (\det M)^{-1}\, d\alpha_1\, d\alpha_2}$

$[\det(MN)]^{-1}\int d\bar{\alpha}d\alpha\, e^{-\bar{\alpha}N^TM\alpha} = 1$

但 $\det MN = \det(M^TN)$ 令 $M^TN = A$

$$\int d\bar{\alpha}\, d\alpha e^{-\bar{\alpha}A\alpha} = \det A$$

无限维 Grassmann 代数生成元 $\varepsilon(x)$

$$\begin{cases} \{\varepsilon(x), \varepsilon(y)\} = 0 \\ \dfrac{\partial^{L,R}\varepsilon(x)}{\partial\varepsilon(y)} = \delta(x-y) \\ \int d\varepsilon(x) = 0\ ;\ \int \varepsilon(x)\, d\varepsilon(x) = 1 \end{cases}$$

Dirac 场的 $\mathcal{L}$

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi$$

$$Z_0[\eta,\bar{\eta}] = \frac{1}{N}\int D\bar{\psi}D\psi \exp\{i\int[\bar{\psi}(x)(i\gamma\cdot\partial - m)\psi(x) + \bar{\eta}\psi(x) + \bar{\psi}(x)\eta(x)]\,dx\}$$

$$N = \int D\bar{\psi}D\psi \exp\left[i\int(\bar{\psi}(x)(i\gamma\cdot\partial - m)\psi(x)\,dx\right]$$

定义 $S^{-1} \equiv i\gamma^\mu\partial_\mu - m$

$$\mathcal{L} = \bar{\psi}S^{-1}\psi$$

令 $Q(\psi,\bar{\psi}) = \bar{\psi}S^{-1}\psi + \bar{\eta}\psi + \bar{\psi}\eta$

Q_{min}, $\psi = ?$

$$\psi_m = -S\bar{\eta}\ ,\ \bar{\psi}_m = -\eta S$$

S^{-1}

$$Q_m = -\bar{\eta}S\eta$$

$$Z_0 = \frac{1}{N}\int D\bar{\psi} D\psi \exp\left\{ i\int [\mathcal{L}_m + (\psi - \psi_m) S^{-1} (\bar{\psi} - \bar{\psi}_m)] dx \right\}$$

$$= \frac{1}{N} \exp\left[-i\int \bar{\eta}(x) S\, \eta(y)\, dx dy \right] \det(-iS^{-1})$$

$$\underbrace{e^{i\mathcal{L}_m}} \quad \psi\bar{\psi}$$

此处 $N = \det(-iS^{-1})$

$$Z_0[\eta, \bar{\eta}] = \exp\left[-i\int \bar{\eta}(x) S(x-y)\, \eta(y)\, dx dy \right]$$

$$S(x) = (i\gamma\cdot\partial + m)\underbrace{\Delta_F(x)}_{\text{Feynman propagator}}$$

$$S^{-1}S = (i\gamma\cdot\partial - m)(i\gamma\cdot\partial + m)\Delta_F(x)$$
$$= [\Box - m^2]\Delta_F(x)$$
$$= \delta^4(x)$$

$$\tau(x,y) = \frac{\delta^2 Z_0[\eta,\bar{\eta}]}{\delta\eta(x)\,\delta\bar{\eta}(y)}\Big|_{\eta=\bar{\eta}=0}$$

$$= -\frac{\delta}{\delta\eta(x)} \cdot \frac{\delta}{\delta\eta(y)} \exp\left[-i\int \bar{\eta}(x) S(x-y)\, \eta(y)\, dx dy \right]\Big|_{\eta=\bar{\eta}=0}$$

$$= iS(x-y)$$

标量场 - 玻色场

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 = -\frac{1}{2}\phi(\Box + m^2)\phi$$

两点 (2-point) 函数

$$\tau(x,y) = i\Delta_F(x-y)$$

此处 Δ_F 满足:

$$(\Box+m^2)\Delta_F(x-y) = -\delta^4(x-y)$$

$$\mathcal{L}_0 = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi = \bar{\psi}S^{-1}\psi$$

两点函数：$\tau(x,y) = iS(x-y)$

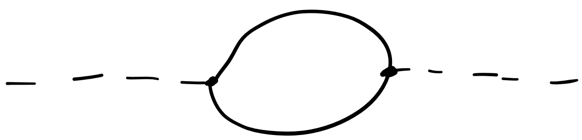
假设了 $\begin{cases} [\xi_i, d\xi_j] = 0 \\ [d\xi_i, d\xi_j] = 0 \end{cases}$

$$\frac{\delta^2}{\delta\eta(x)\delta\eta(y)} = -\frac{\delta^2}{\delta\eta(y)\delta\eta(x)}$$

其中 η 费米场源

左微分

$$\frac{\delta}{\delta\eta(x_1)}[\eta(x)\eta(y)] = \delta^4(x_1-x)\eta(y) - \delta^4(x_1-y)\eta(x)$$



$$Z[\eta,\bar{\eta}] = \exp\left[i\int\mathcal{L}_{int}\left(\frac{1}{i}\frac{\delta}{\delta\eta}, \frac{1}{i}\frac{\delta}{\delta\bar{\eta}}\right)dx\right]Z_0(\eta,\bar{\eta})$$

展开后第三项：

$$-\frac{1}{2}\int dx\,dy\,dx'\,dy'\,\bar{\eta}(x)S(x-y)\eta(y)\bar{\eta}(x')S(x'-y')\eta(y')$$

loop 贡献了：

$$\frac{\delta^2}{\delta\bar{\eta}_i(z)\delta\eta_j(z)}\frac{\delta^2}{\delta\bar{\eta}_k(z')\delta\eta_l(z')}Z(\eta,\bar{\eta})$$

$\Rightarrow + S_{il}(z-z')S_{kj}(z'-z)$

S-matrix

$\alpha \rightarrow \beta$

$S_{\beta\alpha} = \langle \beta, t\to\infty | \alpha, t\to-\infty \rangle$

$|\alpha\rangle_{in} = |\alpha, t\to-\infty\rangle$, $|\beta\rangle_{out} = |\beta, t\to\infty\rangle$

scalar $\vec{p}_1 \cdot \vec{p}_2$

$|\alpha\rangle_{in} = a^\dagger_{in}(\vec{p}_1)\, a^\dagger_{in}(\vec{p}_2)|0\rangle$

$$\begin{cases} a_{out}(p) = S^\dagger a_{in}(p) S \\ a^\dagger_{out}(p) = S^\dagger a^\dagger_{in}(p) S \end{cases}$$

$\phi_{out}(x) = S^\dagger \phi_{in}(x) S$

$\phi(x)$ $\underset{t}{(-\infty, +\infty)}$

$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 + \mathcal{L}_{int}$

ϕ 遵循运动方程:

$$\underbrace{(\Box_x + m^2)}_{K_x}\phi(x) = \frac{\partial\mathcal{L}_{int}}{\partial\phi(x)}$$

$$K_x\phi(x) = \frac{\partial\mathcal{L}_{int}}{\partial\phi(x)}$$

$G(x-y)$

$$(\Box_y + m^2)\, G(y-x) = \delta^4(y-x)$$

$$\Rightarrow \int d^4y\, [G(y-x)(\Box_y + m^2)\phi(y) - \phi(y)(\Box_y + m^2) G(y-x)] = \int G(y-x) \frac{\partial \mathcal{L}_{int}}{\partial \phi(y)} d^4y - \phi(x)$$

LHS

$$-\int d^3y\, dy_0 \left[\underline{[G(y-x)\nabla^2\phi(y) - \phi(y)\nabla^2 G(y-x)]} - \left[G(y-x) \frac{\partial^2}{\partial y_0^2}\phi(y) - \phi(y)\frac{\partial^2}{\partial y_0^2} G(y-x) \right] \right]$$

$$\int d^3y\, dy_0 (G\nabla^2\phi - \phi\nabla^2 G) = \int dS^2\, dy_0 (G\nabla\phi - \phi\nabla G) = 0$$

$$G\frac{\partial^2}{\partial y_0^2}\phi - \phi\frac{\partial^2}{\partial y_0^2}G = \frac{\partial}{\partial y_0}\left(G\frac{\partial}{\partial y_0}\phi - \phi\frac{\partial}{\partial y_0}G\right) = \frac{\partial}{\partial y_0}(G\overleftrightarrow{\partial_0}\phi)$$

$$\phi(x) = -\left(\int_{y_0^+} - \int_{y_0^-}\right) d^3y\, G(y-x)\overleftrightarrow{\partial_0}\phi(y) + \int_{y_0^-}^{y_0^+} dy\, G(y-x)\frac{\partial \mathcal{L}_{int}}{\partial \phi(y)}$$

$$\begin{cases} \Delta_{ret}(x) = 0 & x^2 > 0,\ x_0 < 0 \\ \Delta_{adv}(x) = 0 & x^2 > 0,\ x_0 < 0 \end{cases}$$

$$(\Box + m^2)\Delta_{\substack{ret\\adv}}(x) = \delta^4(x)$$

$G_1 . G_2$

$$\begin{cases} G_1(x-y)\,(\Box_x+m^2)\,G_2(x-z) = \delta(x-z)\,G_1(x-y) \\ G_2(x-z)\,(\Box_x+m^2)\,G_1(x-y) = \delta(x-y)\,G_2(x-z) \end{cases}$$

$$G_1(z-y) - G_2(y-z) = \left(\int_{x_0^+} - \int_{x_0^-}\right) d^3x\, G_2(x-z)\overleftrightarrow{\partial_0} G_1(x-y)$$

$$\Delta_{ret}(x) = \Delta_{adv}(-x)$$

$$G = \Delta_{adv}$$

$$\phi(x) = \int_{y_0^-} d^3y\, \Delta_{ret}(x-y)\overleftrightarrow{\partial_0}\phi(y) + \int dy\, \Delta_{ret}(x-y)\frac{\partial \mathcal{L}_{int}}{\partial\phi(y)}$$

$$\phi_{-\infty}(x) = \lim_{y_0\to-\infty}\int_{y_0} d^3y\, \Delta_{ret}(x-y)\overleftrightarrow{\partial_{y_0}}\phi(y)$$

$$(\Box+m^2)\phi_{-\infty}(x) = \lim_{y_0\to-\infty}\int d^3y\, \delta^4(x-y)\overleftrightarrow{\partial_{y_0}}\phi(y)$$

$$\phi(x) = \phi_{in} + \int dy\, \Delta_{ret}(x-y)\frac{\partial\mathcal{L}_{int}}{\partial\phi(y)}$$

或

$$\phi(x) = \phi_{out} + \int dy\, \Delta_{adv}(x-y)\frac{\partial\mathcal{L}_{int}}{\partial\phi(y)}$$

$$\begin{cases} \phi(x) = \phi_{in}(x) + \int dy\, \Delta_{ret}(x-y)K_y\phi(y) \\ \phi(x) = \phi_{out} + \int dy\, \Delta_{adv}(x-y)K_y\phi(y) \end{cases}$$

$$\phi(x) \xrightarrow[t\to\pm\infty]{} \phi^{out}_{in}(x) \quad \to \text{strong}$$

$$\lim_{t\to+\infty}\langle a|\phi(x)|b\rangle = \langle a|\phi^{as}_{in}(x)|b\rangle$$

$$\phi_{out}(x) = S^{\dagger}\phi_{in}(x)S$$

define:

$$I[J] = T\exp\left[i\int J(x)\phi(x)\,dx\right]$$

$$\Rightarrow \frac{1\,\delta I[J]}{i\,\delta J(x)} = T(\phi(x)I[J])$$

$$Z[J] = \langle 0|I[J]|0\rangle$$

$$\frac{1\,\delta I[J]}{i\,\delta J(x)} = I[J]\phi_{in}(x) + \int dy\,\Delta_{ret}(x-y)K_y\frac{1\,\delta I[J]}{i\,\delta J(x)}$$

$$\frac{1\,\delta I[J]}{i\,\delta J(x)} = \phi_{out}I[J] + \int dy\,\Delta_{adv}(x-y)K_y\frac{1\,\delta I[J]}{i\,\delta J(x)}$$

$$\phi_{out}I - I\phi_{in} = i\int dy\,\Delta(x-y)K_y\frac{\delta I[J]}{\delta J(y)}$$

其中 $\Delta(x) = \Delta_{adv}(x) - \Delta_{ret}(x)$

$$\underline{[\phi_{in}(x), SI[J]] = i\int dy\,\Delta(x-y)K_y\frac{\delta(SI[J])}{\delta J(y)}}$$

$$e^{B}Ae^{-B} = A + [B,A]$$

$$[A, e^{B}] = [A,B]e^{B}$$

$$\langle 0|\quad|p\rangle$$

$$\langle 0| K_y \phi(y) |p\rangle = \langle 0| (\Box+m^2) \phi(y) |p\rangle$$
$$= (\Box+m^2) e^{-ipy} \langle 0| \phi(0) |p\rangle$$
$$= (m^2-p^2) \langle 0| \phi(y) |p\rangle$$
$$= 0$$

$$\langle 0| [\phi_{in}(x), \phi_{in}(y)] |0\rangle = \langle 0| [\phi(x), \phi(y)] |0\rangle = i\Delta(x-y)$$

$$SI[J] = \exp\left[\int \phi_{in}(z) K \frac{\delta}{\delta J(z)} dz\right] F[J]$$

$$[\phi_{in}(x), SI] = i\int \Delta(x-y) K \frac{\delta}{\delta J(y)} dy \exp\left[\int \phi_{in}(z) K \frac{\delta}{\delta J(z)} dz\right] F[J]$$
$$= \exp\left[\int \phi_{in}(z) K \frac{\delta}{\delta J(z)} dz\right] i\int \Delta(x-y) K \frac{\delta F[J]}{\delta J(y)} dy$$

$$\frac{\delta(SI[J])}{\delta J(y)} = \exp\left[\int \phi_{in}(z) K \frac{\delta}{\delta J(z)} dz\right] \frac{\delta F[J]}{\delta J(y)}$$

$$\langle 0| :e^A: |0\rangle = 1$$

$$\langle 0| SI[J] |0\rangle = F[J]$$

无外场，令 $\langle 0| S = \langle 0|$

$$\langle 0| SI[J] |0\rangle = \langle 0| I[J] |0\rangle = Z[J]$$

$$SI[J] = :\exp\left[\int \phi_{in}(z) K \frac{\delta}{\delta J(z)} dz\right]: Z[J]$$

$$\left(\frac{\delta}{\delta J}\right)^n Z[J]\Big|_{J=0}$$

$$\frac{1}{i}\frac{\delta}{\delta J(x_1)}\frac{1}{i}\frac{\delta}{\delta J(x_2)}\cdots\frac{1}{i}\frac{\delta}{\delta J(x_n)} Z[J]\Big|_{J=0} = G(x_1,\dots,x_n)$$

$$\prod_i (\Box_{x_i} + m^2)$$

$$\prod_i \phi(x_i)$$

$$S_n(x_1,\dots,x_n) = \prod_i \phi(x_i)(\Box_{x_i}+m) G(x_1,\dots,x_n)$$